Master Parisien de Recherche en Informatique

Course 2.16 – Finite automata based computation models

6 march 2019 — Exam (2) — Part B

Books and computers forbidden — Lecture and personal notes allowed.
This part should be written on separate test papers.

Markov chains and linear dynamical systems

(1–a) Build a Markov chain $\mathcal{M}$ such that $\mathbb{P}_{\mathcal{M}}(n) = 2^{-n}$ for all $n \in \mathbb{N}$.

(1–b) Consider the Markov chain $\mathcal{M} = \langle S, M, T\rangle$ defined by

$$S = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Show that $\mathbb{P}_{\mathcal{M}}(n) = 2^{-n}n$ for all $n \in \mathbb{N}$.

(1–c) Build a Markov chain $\mathcal{M}$ such that $\mathbb{P}_{\mathcal{M}}(n) = 2^{-n}\frac{n(n-1)}{2}$ for all $n \geqslant 1$ and 0 otherwise.

(1–d) Show that for any $p \in \mathbb{N}$, there exists a Markov chain $\mathcal{M}_p$ such that $\mathbb{P}_{\mathcal{M}_p}(n) = 2^{-n}\binom{n}{p}$ for all $n \in \mathbb{N}$, where we recall that $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ if $p \leqslant n$ and 0 otherwise, is a binomial coefficient.

(1–e) Show any for any polynomial p with rational coefficients, there exists $d \in \mathbb{N}$ and $a_0, \dots, a_d \in \mathbb{Q}$ such that $p(n) = \sum_{i=0}^{d} a_i \binom{n}{i}$ for all $n \in \mathbb{N}$.

(1–f) Let $\mathcal{M}_0, \dots, \mathcal{M}_d$ be d Markov chains, and $\alpha_0, \dots, \alpha_d \in [0, 1]$ be such that $\alpha_0 + \cdots + \alpha_d \leqslant 1$. Then show that there a Markov chain $\mathcal{N}$ such that

$$\mathbb{P}_{\mathcal{N}}(n) = \alpha_0\mathbb{P}_{\mathcal{M}_0}(n) + \cdots + \alpha_d\mathbb{P}_{\mathcal{M}_d}(n).$$

(1–g) Show that for any polynomial p with rational coefficients, there exists some nonzero constant $\beta \in \mathbb{Q}$ and a Markov chain $\mathcal{M}$ such that $\mathbb{P}_{\mathcal{M}}(n) = \frac{1}{2} + 2^{-n}\beta p(n)$ for all $n \in \mathbb{N}$. *Hint: start with the case where all a_i are nonnegative in (1–e).*

Probabilistic automata: one undecidability result to rule them all

Recall that the value of a probabilistic automaton $\mathcal{A}$ is $\mathrm{val}(\mathcal{A}) = \sup\{\mathbb{P}_{\mathcal{A}}(w) : w \in A^*\}$. The goal of this exercise is to show the following result and see why it subsumes several classical theorems.

Theorem 1. *There is no algorithm such that given a probabilistic automaton $\mathcal{A}$,*

- *if* $\mathrm{val}(\mathcal{A}) = 1$*, then the algorithm outputs "yes",*
- *if* $\mathrm{val}(\mathcal{A}) \leqslant \frac{1}{2}$*, then the algorithm outputs "no",*
- *otherwise, the algorithm can output anything or not terminate.*

(2–a) In the course, we have shown that the following problem (known as "value 1") is undecidable: given $\mathcal{A}$, decide whether $\mathrm{val}(\mathcal{A}) = 1$. Explain why Theorem 1 implies this result from the course.

Let $A = \{0, 1\}$, given a word $w \in A^*$, we define its binary encoding by $[w] = \sum_{i=1}^{|w|} w_i 2^{-i} \in [0, 1]$.

(2–b) Show that for any word $w \in A^*$, $[w0] = [w]$ and $[w1] = [w] + 2^{-|w|-1}$.

(2–c) Consider automata $\mathcal{A}$ from Figure 1a: give its complete description $\langle A, Q, S, \mu, T\rangle$ and show (by induction) that it satisfies $\mathbb{P}_{\mathcal{A}}(w) = [w]$ for all $w \in \{0,1\}^*$.

(2–d) Let $x \in [0,1]$ and consider automaton $\mathcal{B}_x$ from Figure 1b. Show that for any $n \in \mathbb{N}$,

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} L\right) = \tfrac{1}{2}x^n \quad \text{and} \quad \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} R\right) = \tfrac{1}{2}(1-x)^n.$$

We fix an integer N and now analyze the outcome of reading $(\texttt{check}\cdot\texttt{sim}^n)^N$. After reading $\texttt{check}\cdot\texttt{sim}^n$ from p, the automaton can in states p, L or R.

(2–e) Compute the probability of staying in p, that is $\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} p\right)$.

(2–f) What happens when reading $\texttt{check}\cdot\texttt{sim}^n$ from L ?

(2–g) What happens when reading $\texttt{check}\cdot\texttt{sim}^n$ from R ?

(2–h) Show that

$$\mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^{N+1}) = \mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^{N}) + \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^N} L\right).$$

(2–i) Show that

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^{N+1}} L\right) = \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^{N}} p\right)\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} L\right).$$

(2–j) Show that

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^{N}} p\right) = \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} p\right)^N.$$

(2–k) Let $p_n = \frac{1}{2}x^n$ and $q_n = \frac{1}{2}(1-x)^n$. Show that

$$\mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^{N}) = \frac{1}{1+\frac{q_n}{p_n}}\left(1-(1-p_n-q_n)^{N-1}\right).$$

We now let $N = 2^n$ and assume that $x > \frac{1}{2}$.

(2–l) Show that $\frac{q_n}{p_n}$ and $(1-p_n-q_n)^{N-1}$ converges to 0, when n tends to infinity.

(2–m) What is the value of $\mathcal{B}_x$ (when $x > \frac{1}{2}$) ?

Let $\mathcal{C}$ be any probabilistic automaton on some alphabet A, which we assume (without loss of generality) to only have one initial state q_0 that is not accepting, and consider automaton $\mathcal{D}$ on alphabet $\Sigma = A \cup \{\texttt{check}, \texttt{end}\}$ from Figure 2. The transitions coming out of $\mathcal{C}$ are from the accepting states of $\mathcal{C}$, the *dashed* transitions coming out of $\mathcal{C}$ are from the *non-accepting* the states, the *dotted* transitions coming out of $\mathcal{C}$ are only from q_0. We rename the state q_0 to L in $\mathcal{C}_l$ and to R in $\mathcal{C}_R$.

(2–n) Let $w \in A^*$, describe the possible outcomes when reading $w\cdot\texttt{end}$ from p, L, R, q_L and q_R and theirs probabilities. Show that $w\cdot\texttt{end}$ has the same transition probabilities as $\texttt{sim}$ in $\mathcal{B}_x$ where $x = \mathbb{P}_{\mathcal{C}}(w)$.

(2–o) Show that if $\mathbb{P}_{\mathcal{C}}(w) > \frac{1}{2}$ then $\mathrm{val}(\mathcal{D}) = 1$.

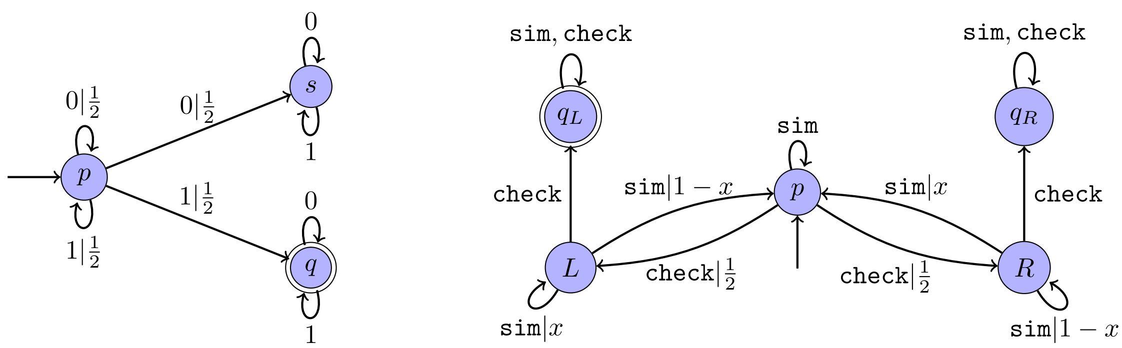
(2–p) Let $w \in \Sigma^*$, show that

$$\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} q_L\right) \leqslant \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} q_R\right) \quad \text{and} \quad \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} L\right) \leqslant \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} R\right)$$

by induction by considering the following cases: $w = w'\cdot\texttt{check}\cdot A^*\cdot\texttt{end}$ $w = w'\cdot\texttt{check}\cdot A^*$ and $w \in (A\cup\{\texttt{end}\})^*$. Explain why this case distinction is exhaustive.

(2–q) Show that if $\mathbb{P}_{\mathcal{C}}(w) \leqslant \frac{1}{2}$ for all $w \in A^*$ then $\mathrm{val}(\mathcal{D}) \leqslant \frac{1}{2}$.

(2–r) Using the fact that the emptiness problem for stochastic languages is undecidable, show Theorem 1.



(a) Automaton $\mathcal{A}$.

(b) Automaton $\mathcal{B}_x$, with $x \in [0, 1]$.

Figure 1: Probabilistic automata for the proof of Theorem 1.

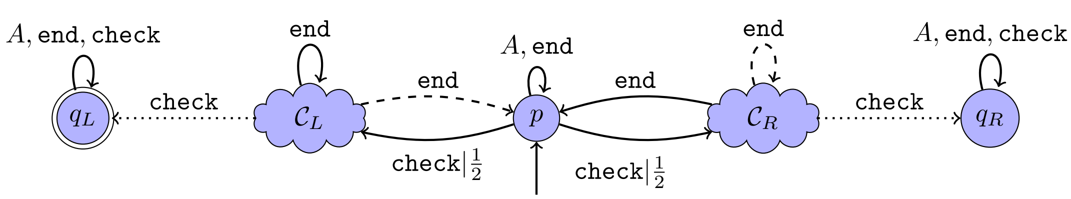


Figure 2: Probabilistic automaton $\mathcal{D}$ for the proof of Theorem 1.

Solutions to exercises

(1–a) Let

$$S = \begin{bmatrix}1 & 0\end{bmatrix}, \qquad M = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} \\ 0 & 1\end{bmatrix}, \qquad T = \begin{bmatrix}1 \\ 0\end{bmatrix}.$$

Then for all $n \in \mathbb{N}$,

$$SM^nT = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2^n} & 1-\frac{1}{2^n} \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = 2^{-n}.$$

(1–b) We will show by induction that

$$SM^n = \begin{bmatrix}2^{-n} & 2^{-n}n & 1-2^{-n}(1+n)\end{bmatrix}$$

Indeed ($*$ denotes anything),

$$\begin{bmatrix}2^{-n} & 2^{-n}n & *\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}2^{-n} & \frac{1}{2}2^{-n}+\frac{1}{2}2^{-n}n & *\end{bmatrix} = \begin{bmatrix}2^{-n-1} & 2^{-n-1}(n+1) & *\end{bmatrix}.$$

(1–c) Let

$$S = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \qquad M = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1\end{bmatrix}, \qquad T = \begin{bmatrix}0 \\ 0 \\ 1 \\ 0\end{bmatrix}.$$

Then for all $n \in \mathbb{N}$, we will show by induction that

$$SM^n = \begin{bmatrix}2^{-n} & 2^{-n}n & 2^{-n}\frac{n(n-1)}{2} & *\end{bmatrix}.$$

Indeed ($*$ denotes anything),

$$\begin{bmatrix}2^{-n} & 2^{-n}n & 2^{-n}\frac{n(n-1)}{2} & *\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}2^{-n} & \frac{1}{2}2^{-n}+2^{-n}n & \frac{1}{2}2^{-n}n+\frac{1}{2}2^{-n}\frac{n(n-1)}{2} & *\end{bmatrix}$$

and conclude by noting $n + \frac{n(n-1)}{2} = \frac{2n+n^2-n}{2} = \frac{n(n+1)}{2}$.

(1–d) Consider the following Markov chain in dimension $p+2$:

$$S = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}, \qquad M = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1\end{bmatrix}, \qquad T = \begin{bmatrix}0 \\ \vdots \\ 1 \\ 0\end{bmatrix}.$$

Then check by induction that

$$SM^n = 2^{-n}\begin{bmatrix}\binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{p} & *\end{bmatrix}.$$

Indeed,

$$2^{-n}\begin{bmatrix}\binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{p} & *\end{bmatrix}M = 2^{-n}\begin{bmatrix}\frac{1}{2}\binom{n}{0} & \frac{1}{2}\binom{n}{0}+\frac{1}{2}\binom{n}{1} & \cdots & \frac{1}{2}\binom{n}{p-1}+\binom{n}{p} & *\end{bmatrix}$$

and conclude using Pascal's rule: $\binom{n}{p-1}+\binom{n}{p} = \binom{n+1}{p}$.

(1–e) Proceed by induction on the degree of p: if $p(x) = a_0$ is constant then $p(n) = a_0\binom{n}{0}$ for all $n \in \mathbb{N}$. Otherwise, write $p(x) = a_dx^d + r(x)$ where q has degree at most $d-1$ and consider $q(x) = p(x) - a_d\frac{x(x-1)\cdots(x-d+1)}{d!}$. Then q has degree at most $d-1$ because

$$a_d\frac{x(x-1)\cdots(x-d+1)}{d!} = a_dx^d + \text{monomials of degree less than } d$$

has the same leading monomial. By induction, there exists $a_0, \ldots, a_{d-1}$ such that $q(n) = \sum_{i=0}^{d-1} a_i\binom{n}{i}$ for all $n \in \mathbb{N}$. Then $p(n) = \sum_{i=0}^{d} a_i\binom{n}{i}$ for all $n \in \mathbb{N}$.

(1–f) The automaton below answers the question:

(1–g) Let $a_0, \dots, a_d$ be as in (1–e). For each $i \in \{0, \dots, d\}$, write $a_i = a_i^+ - a_i^-$ where $a_i^+, a_i^- \geqslant 0$ and let $\beta = \max(a_0^+ + \cdots + a_d^+, a_0^- + \cdots + a_d^-)$. For each i, let $\mathcal{M}_i$ be such that $\mathcal{M}_i(n) = 2^{-n}\binom{n}{i}$. Then let $\mathcal{N}^+$ be such that $\mathbb{P}_{\mathcal{N}^+}(n) = \sum_{i=0}^{d} \frac{a_i^+}{\beta}\mathcal{M}_i(n)$ for all n, and similarly for $\mathcal{N}^-$. This is possible because $\frac{a_i^+}{\beta} \in [0,1]$ and $\sum_{i=0}^{d} \frac{a_i^+}{\beta} = \frac{a_0^+ + \cdots + a_d^+}{\beta} \leqslant 1$. Then build $\mathcal{N}_c^-$ such that $\mathbb{P}_{\mathcal{N}_c^-}(n) = 1 - \mathbb{P}_{\mathcal{N}^-}(n)$ and finally $\mathcal{N}$ such that $\mathbb{P}_{\mathcal{N}}(n) = \frac{1}{2}\mathbb{P}_{\mathcal{N}^+}(n)\frac{1}{2}\mathbb{P}_{\mathcal{N}_c^-}(n)$. Putting everything together, we get

$$
\begin{aligned}
\mathbb{P}_{\mathcal{N}}(n) &= \tfrac{1}{2} + \tfrac{1}{2}\left(\mathbb{P}_{\mathcal{N}^+}(n) - \mathbb{P}_{\mathcal{N}^-}(n)\right) && \text{by definition of } \mathcal{N}_c^- \\
&= \tfrac{1}{2} + \tfrac{1}{2}\left(\sum_{i=0}^{d} \tfrac{a_i^+}{\beta}\mathbb{P}_{\mathcal{M}_i}(n) - \sum_{i=0}^{d} \tfrac{a_i^-}{\beta}\mathbb{P}_{\mathcal{M}_i}(n)\right) && \text{by definition of } \mathcal{N}^\pm \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sum_{i=0}^{d} \tfrac{a_i^+ - a_i^-}{\beta} 2^{-n}\binom{n}{i} && \text{by definition of } \mathcal{M}_i \\
&= \tfrac{1}{2} + \tfrac{1}{2\beta}2^{-n}\sum_{i=0}^{d} a_i\binom{n}{i} \\
&= \tfrac{1}{2} + \tfrac{1}{2\beta}2^{-n}p(n) && \text{by definition of the } a_i.
\end{aligned}
$$

(2–a) Assume that the "value 1" was decidable: then there is an algorithm that outputs "yes" when $\mathrm{val}(\mathcal{A}) = 1$ and "no" when $\mathrm{val}(\mathcal{A}) < 1$. But Theorem 1 says that such an algorithm does not exist. Thus the "value 1" problem is undecidable.

(2–b) For any $a \in A$, $[wa] = \sum_{i=1}^{|w|} w_i 2^{-i} + a2^{-|w|-1} = [w] + a2^{-|w|-1}$.

(2–c) Let $Q = \{p, q, s\}$ and

$$
S = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad \mu(0) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mu(1) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
$$

We can prove the result either by matrix computation or by reasoning on the automaton:

- Since p is not accepting, $\mathbb{P}_{\mathcal{A}}(\varepsilon) = 0$. Note that for any word w, $\mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{w} p\right) = 2^{-|w|}$ since s and q are sinks. Then, since only q is accepting and there are no transitions between s and q, for any word w and letter a we have

$$
\mathbb{P}_{\mathcal{A}}(wa) = \mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{w} p\right)\mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{a} q\right) + \mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{w} q\right)\mathbb{P}_{\mathcal{A}}\left(q \xrightarrow{a} q\right) = 2^{-|w|}\mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{a} q\right) + \mathbb{P}_{\mathcal{A}}(w).
$$

  It follows that $\mathbb{P}_{\mathcal{A}}(w0) = \mathbb{P}_{\mathcal{A}}(w)$ since $\mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{0} q\right) = 0$ and $\mathbb{P}_{\mathcal{A}}(w1) = 2^{-|w|-1} + \mathbb{P}_{\mathcal{A}}(w)$ since $\mathbb{P}_{\mathcal{A}}\left(p \xrightarrow{1} q\right) = \frac{1}{2}$. By induction, this proves that $\mathbb{P}_{\mathcal{A}}(w) = [w]$.

- We can then check that $S\mu(w) = \begin{bmatrix} 2^{-|w|} & [w] & 1 - [w] - 2^{-|w|} \end{bmatrix}$ by induction:

$$
\begin{aligned}
\begin{bmatrix} 2^{-|w|} & [w] & 1 - [w] - 2^{-|w|} \end{bmatrix}\mu(0) &= \begin{bmatrix} 2^{-|w|-1} & [w] & 1 - [w] - 2^{-|w|} + 2^{-|w|-1} \end{bmatrix} \\
&= \begin{bmatrix} 2^{-|w|-1} & [w0] & 1 - [w0] - 2^{-|w|-1} \end{bmatrix} && \text{since } [w0] = [w]
\end{aligned}
$$

  and

$$
\begin{aligned}
\begin{bmatrix} 2^{-|w|} & [w] & 1 - [w] - 2^{-|w|} \end{bmatrix}\mu(1) &= \begin{bmatrix} 2^{-|w|-1} & [w] + 2^{-|w|-1} & 1 - [w] - 2^{-|w|} \end{bmatrix} \\
&= \begin{bmatrix} 2^{-|w|-1} & [w1] & 1 - [w1] - 2^{-|w|-1} \end{bmatrix} && \text{since } [w1] = [w] + 2^{-|w|-1}
\end{aligned}
$$

  and therefore, $S\mu(w)T = [w]$.

(2–d) Clearly L and R are symmetric in the automaton (by replacing x by $1-x$) so we prove it for L. After reading `check`, the automaton can be in state L or R. But there is no path labelled by $\texttt{sim}^*$ from R to L. Once in L, reading `sim` can make the automaton stay in L or go back to p. But again there is no path labelled by $\texttt{sim}^*$ from p to L. Therefore the only path from p to L with positive probability goes to L first and then stays in L. In other words,

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} L\right) = \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}} L\right)\mathbb{P}_{\mathcal{B}_x}\left(L \xrightarrow{\texttt{sim}} L\right)^n = \tfrac{1}{2}x^n.$$

(2–e) By stochasticity,

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} p\right) = 1 - \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} L\right) - \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} R\right) = 1 - \tfrac{1}{2}x^n - \tfrac{1}{2}(1-x)^n.$$

(2–f) When reading $\texttt{check}\cdot\texttt{sim}^n$ from L, the word is accepted with probability 1.

(2–g) When reading $\texttt{check}\cdot\texttt{sim}^n$ from R, the word is rejected, *i.e.* accepted with probability 0.

(2–h) After reading $(\texttt{check}\cdot\texttt{sim}^n)^N$, the automaton can be in any state, but the only states that lead to an accepting state when reading $\texttt{check}\cdot\texttt{sim}$ are L and q_L. When reading $\texttt{check}\cdot\texttt{sim}^n$ from either, it is accepted with probability 1, therefore

$$\mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^{N+1}) = \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^N} q_L\right) + \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^N} L\right).$$

(2–i) The only state from which L is reacheable by reading $\texttt{check}\cdot\texttt{sim}^n$ is p. Therefore

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^{N+1}} L\right) = \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^N} p\right)\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} L\right).$$

(2–j) The only state from which p is reacheable by reading $\texttt{check}\cdot\texttt{sim}^n$ is p. Therefore

$$\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^{N+1}} p\right) = \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^N} p\right)\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} p\right)$$

and the result follows by induction since $\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\varepsilon} p\right) = 1$.

(2–k)

$$\begin{aligned}
\mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^N) &= \sum_{i=1}^{N-1} \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^i} L\right) && \text{by (2–h)}\\
&= \sum_{i=1}^{N-1} \mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{(\texttt{check}\cdot\texttt{sim}^n)^{i-1}} p\right)\mathbb{P}_{\mathcal{B}_x}\left(p \xrightarrow{\texttt{check}\cdot\texttt{sim}^n} L\right) && \text{by (2–i)}\\
&= \sum_{i=1}^{N-1} (1-p_n-q_n)^{i-1}p_n && \text{by (2–j)}\\
&= p_n \sum_{i=0}^{N-2} (1-p_n-q_n)^i\\
&= p_n \tfrac{1-(1-p_n-q_n)^{N-1}}{1-(1-p_n-q_n)}\\
&= \tfrac{p_n}{p_n+q_n}\left(1-(1-p_n-q_n)^{N-1}\right)\\
&= \tfrac{1}{1+\frac{q_n}{p_n}}\left(1-(1-p_n-q_n)^{N-1}\right).
\end{aligned}$$

(2–l) Since $\frac{1-x}{x} < 1$, $\frac{q_n}{p_n} = \left(\frac{1-x}{x}\right)^n \to 0$ as $n\to\infty$. Similarly, $1-p_n-q_n < 1-\frac{1}{2}x^n \leqslant 1-\frac{1}{2}$ for all n. Therefore $(1-p_n-q_n)^{N-1} \leqslant (1-\frac{1}{2})^{2^n-1} \to 0$ as $n\to\infty$.

(2–m) By definition, $\text{val}(\mathcal{B}_x) \geqslant \mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^{2^n})$ for all $n\in\mathbb{N}$. But we have that $\mathbb{P}_{\mathcal{B}_x}((\texttt{check}\cdot\texttt{sim}^n)^{2^n}) \to 1$ as $n\to\infty$ therefore, $\text{val}(\mathcal{B}_x) = 1$.

(2–n) When reading $w\cdot\texttt{end}$

- from p, q_L, q_R: we stay in this state with probability 1,
- from L: we stay in L with probability x and go to p with probability $1-x$,
- from R: we stay in R with probability $1-x$ and go to p with probability x.

We observe that this is the same transition table as $\texttt{sim}$ in $\mathcal{B}_x$.

(2–o) We have shown in (2–m) that if $x > \frac{1}{2}$ then $\mathrm{val}(\mathcal{B}_x) = 1$. Specifically, $\mathbb{P}_{\mathcal{B}_x}((\texttt{check} \cdot \texttt{sim}^n)^{2^n}) \to 1$ as $n \to \infty$. But we have observed in the last question that $\mathbb{P}_{\mathcal{B}_x}((\texttt{check} \cdot \texttt{sim}^n)^{2^n}) = \mathbb{P}_{\mathcal{D}}((\texttt{check} \cdot (w \cdot \texttt{end})^n)^{2^n})$ since the transition table is the same for $\texttt{sim}$ (and is obviously the same for other letters). Therefore $\mathrm{val}(\mathcal{D}) = 1$.

(2–p) If $w \in (A \cup \{\texttt{end}\})^*$, then the automaton is always in p, thus all other probabilities are 0 and the inequalities hold. Note that this covers the initial induction step ($w = \varepsilon$). Otherwise, w must contain at least one $\texttt{check}$ and it either finishes by $\texttt{end}$ or by a (possibly empty) word in A^*:

- if $w = w' \cdot \texttt{check} \cdot u$ with $u \in A^*$ then after reading $w' \cdot \texttt{check}$ the automaton must be in state L, q_L, R or q_R. Furthermore, for any $s, t \in \{L, R, q_L, q_R\}$, if $s \neq t$ then there is no transition from s to t labelled by u, *i.e.* $\mathbb{P}_{\mathcal{D}}\left(s \xrightarrow{u} t\right) = 0$. Therefore, for any $s \in \{L, R, q_L, q_R\}$, $\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} s\right) = \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w' \cdot \texttt{check}} s\right) \mathbb{P}_{\mathcal{D}}\left(s \xrightarrow{u} s\right)$. Therefore

$$\begin{aligned}
\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} q_L\right) &= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w' \cdot \texttt{check}} q_L\right) && \text{since } \mathbb{P}_{\mathcal{D}}\left(q_L \xrightarrow{u} q_L\right) = 1 \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} L\right) && \text{since } \texttt{check} \text{ comes from } L \text{ only} \\
&\leqslant \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} R\right) && \text{by induction} \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} q_R\right) && \text{by a symmetric reasoning.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} L\right) &= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w' \cdot \texttt{check}} L\right) \mathbb{P}_{\mathcal{C}}\left(q_0 \xrightarrow{u} q_0\right) && \text{since } \mathbb{P}_{\mathcal{D}}\left(L \xrightarrow{u} L\right) = \mathbb{P}_{\mathcal{C}}\left(q_0 \xrightarrow{u} q_0\right) \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) \mathbb{P}_{\mathcal{C}}\left(q_0 \xrightarrow{u} q_0\right) && \text{since } \texttt{check} \text{ comes from } p \text{ only} \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} R\right) && \text{by a symmetric reasoning.}
\end{aligned}$$

- if $w = w' \cdot \texttt{check} \cdot u \cdot \texttt{end}$ with $u \in A^*$ then after reading $w' \cdot \texttt{check}$ the automaton must be in state L, q_L, R or q_R. The analysis for q_L and q_R is the same because there are no transitions from L or R to q_L or q_R labelled by $u \cdot \texttt{end}$. The analysis for L and R is a bit different: note that for L to be reachable w, the automaton must be in state p when reading $\texttt{check} \cdot u \cdot \texttt{end}$ and similarly for R. Therefore

$$\begin{aligned}
\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} L\right) &= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{\texttt{check} \cdot u \cdot \texttt{end}} L\right) && \text{by the above remark} \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{\texttt{check}} L\right) \mathbb{P}_{\mathcal{D}}\left(L \xrightarrow{u \cdot \texttt{end}} L\right) \\
&= \tfrac{1}{2} \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) \mathbb{P}_{\mathcal{C}}(u) && \text{since } \texttt{end} \text{ goes to } L \text{ only from accepting states} \\
&\leqslant \tfrac{1}{2} \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) (1 - \mathbb{P}_{\mathcal{C}}(u)) && \text{since } \mathbb{P}_{\mathcal{C}}(u) \leqslant \tfrac{1}{2} \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{\texttt{check}} R\right) \mathbb{P}_{\mathcal{D}}\left(R \xrightarrow{u \cdot \texttt{end}} R\right) && \text{since } \texttt{end} \text{ goes to } R \text{ from non-accepting states} \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w'} p\right) \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{\texttt{check} \cdot u \cdot \texttt{end}} R\right) \\
&= \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} R\right) && \text{by the above remark.}
\end{aligned}$$

(2–q) By the previous question, we get that $\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} L\right) \leqslant \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} R\right)$ but by stochasticity $\mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} L\right) + \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} R\right) \leqslant 1$, therefore $\mathbb{P}_{\mathcal{D}}(w) = \mathbb{P}_{\mathcal{D}}\left(p \xrightarrow{w} L\right) \leqslant \frac{1}{2}$ for every word $w \in \Sigma^*$.

(2–r) Assume there was an algorithm X as described in Theorem 1. Then consider the algorithm (call it Y) that given an automaton $\mathcal{C}$ as input, builds the automaton $\mathcal{D}$ and runs X on it. If $\mathcal{L}_{\mathcal{C}}^{>}(\frac{1}{2}) = \varnothing$ then $\mathbb{P}_{\mathcal{C}}(w) \leqslant \frac{1}{2}$ for all words w therefore $\operatorname{val}(\mathcal{D}) \leqslant \frac{1}{2}$ by the previous question and therefore X will output "no" on $\mathcal{D}$. Conversely, if $\mathcal{L}_{\mathcal{C}}^{>}(\frac{1}{2}) \neq \varnothing$ then $\mathbb{P}_{\mathcal{C}}(w) > \frac{1}{2}$ for some word w therefore $\operatorname{val}(\mathcal{D}) = 1$ by (2–o) and therefore X will output "yes" on $\mathcal{D}$. But then algorithm Y decides the emptiness of stochastic languages which is a contradiction.