Master Parisien de Recherche en Informatique

Course 2.16 – Finite automata based computation models

4 december 2020 — Exam (1) — Part A

Internet searches forbidden — Lecture and personal notes allowed.

The two parts are independent and can be done in any order.

Notations. For any set X of words, X^* denotes the Kleene star of X and $X^+ = XX^*$ denotes the Kleene plus of X. For any vector or matrix A, A^T denotes its transpose. For any $n \in \mathbb{N}$, $\mathbf{1}_n$ is the column vector of dimension n consisting of all ones. We simply write $\mathbf{1}$ when the dimension is clear. For any vector v, we let $v^c = \mathbf{1} - v$ be the complement of v. For any $n, m \in \mathbb{N}$, $\mathbf{0}_{n,m}$ denotes the $n \times m$ matrix consisting of all zeroes. We simply write $\mathbf{0}$ when the dimensions are clear. For any set $X \subseteq \mathbb{R}^n$, $\overline{X}$ denotes the topological closure of X (smallest closed set containing X).

1 Counting with stochastic language

Let $\Sigma = \{a\}$ be a unary alphabet. For any $n \in \mathbb{N}$, let $C_n = \{a^n\}$ be the language consisting of a single word a^n.

(a1) By using Myhill-Nerode theorem, show that for any $n \in \mathbb{N}$, the smallest deterministic complete finite automaton recognizing C_n has exactly $n + 2$ states.

(a2) Let $\delta \in [0, 1]$, build a probabilistic automaton $\mathcal{A}$ with 3 states (including any sink state) such that $\mathcal{A}(\varepsilon) = 0$ and $\mathcal{A}(a^\ell) = (1 - \delta)^{\ell-1}\delta$ for any $\ell \geqslant 1$.

(a3) Modify your automaton (still with 3 states) so that $\mathcal{A}(a^\ell) = (1 - \delta)^{\ell-1}\delta\ell$ for any $\ell \geqslant 1$.

(a4) Show that for any $n \in \mathbb{N}$, there exists a choice of δ such that $\ell \mapsto \mathcal{A}(a^\ell)$ has unique maximum at $\ell = n$.

(a5) Show that for any $n \in \mathbb{N}$, the language C_n is recognized by a 3−state probabilistic automaton with an isolated cut-point. What is your conclusion?

2 Existence of a cut-point in a probabilistic automata

Let $u \in [0, \frac{1}{4}]$ and let $D_u \subseteq [0, 1]$ be the smallest set such that $0 \in D_u$ and if $x \in D_u$ then $f_i(u, x) \in D_u$ for all $i \in \{0, 1, 2, 3\}$, where $f_i(u, x) := \frac{1-u}{3} i + ux$.

(b1) Show that D_u is dense in $[0, 1]$ if $u = \frac{1}{4}$.

(b2) Show that D_u is not dense in $[0, 1]$ if $u < \frac{1}{4}$. *Hint: show that $D_u \subseteq [0, \frac{1}{4} - \varepsilon] \cup [\frac{1}{4}, 1]$ for some $\varepsilon > 0$.*

Consider the probabilistic automaton $\mathcal{A}_u = (A, Q, S, T, \mu)$ where $A = \{0, 1, 2, 3\}$, $Q = \{1, 2\}$, $S = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b3) Explain how to choose μ so that $\{\mathcal{A}_u(w) : w \in A^*\} = D_u$.

(b4) Show that $\mathcal{A}_u$ has an isolated cut-point if and only if $u < \frac{1}{4}$.

Let $\mathcal{B},\mathcal{C}$ be two arbitrary probabilistic automata over some alphabet Σ. Write $\mathcal{B}=(\Sigma,Q_1,D_1,T_1,\mu_1)$ and $\mathcal{C}=(\Sigma,Q_2,D_2,T_2,\mu_2)$. We consider the automaton $\mathcal{D}=(A',Q',S',T',\mu')$ where $\Sigma'=\Sigma\cup\{\sharp\}$ for some fresh $\sharp\notin\Sigma$, $Q'=Q_1\cup Q_2$,

$$S'=\tfrac{1}{\alpha}\begin{bmatrix}(T_1^c)^T & (T_2^c)^T\end{bmatrix},\qquad \mu'(\sigma)=\begin{bmatrix}\mu_1(\sigma) & \mathbf{0}\\ \mathbf{0} & \mu_2(\sigma)\end{bmatrix},\qquad \mu'(\sharp)=\begin{bmatrix}T_1^c & T_1\\ T_2^c & T_2\end{bmatrix}\begin{bmatrix}D_1 & \mathbf{0}\\ \mathbf{0} & D_2\end{bmatrix}=\begin{bmatrix}T_1^cD_1 & T_1D_2\\ T_2^cD_1 & T_2D_2\end{bmatrix},\qquad T'=\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}$$

and $\alpha=(T_1^c+T_2^c)^T\mathbf{1}$ is such that S' is stochastic.

(b5) Show that $\mu'(\sharp)$ is stochastic. Show that for all $k\geqslant 0$ and $w^{(1)},\dots,w^{(k)}\in\Sigma^*$, we have

$$\mathcal{D}(\sharp w^{(1)}\sharp w^{(2)}\sharp\cdots\sharp w^{(k)}\sharp)=\begin{bmatrix}1 & 0\end{bmatrix}\prod_{i=1}^{k}M(w^{(i)})\begin{bmatrix}0\\1\end{bmatrix},\qquad\text{where}\quad M(w):=\begin{bmatrix}1-\mathcal{B}(w) & \mathcal{B}(w)\\ 1-\mathcal{C}(w) & \mathcal{C}(w)\end{bmatrix}\qquad\forall w\in\Sigma^*.$$

Let $\mathcal{B}_i$ and $\mathcal{C}_i$, for $i\in A=\{0,1,2,3\}$, be arbitrary automata on some alphabet Γ such that $A\cap\Gamma=\varnothing$.

(b6) Show that there exist automata $\mathcal{B}$ and $\mathcal{C}$ on alphabet $\Sigma=A\cup\Gamma$ such that for any word $w\in\Gamma^*$ and $i\in A$ we have $\mathcal{B}(iw)=\mathcal{B}_i(w)$ and $\mathcal{C}(iw)=\mathcal{C}_i(w)$, and for any word $w'\notin A\Gamma^*$, we have $\mathcal{B}(w')=\mathcal{C}(w')=0$.

Let $\mathcal{E}$ be an arbitrary probabilistic automaton on alphabet Γ.

(b7) Show that there exist automata $\mathcal{B}_i$ and $\mathcal{C}_i$ on alphabet Γ such that for all $i\in A$ and any word $w\in\Gamma^*$, we have

$$\mathcal{B}_i(w)=\tfrac{1-\mathcal{E}(w)}{3}i,\qquad \mathcal{C}_i(w)=\mathcal{E}(w)+\mathcal{B}_i(w).$$

(b8) Show that with this choice of $\mathcal{B}_i$ and $\mathcal{C}_i$ (and hence of $\mathcal{B}$ and $\mathcal{C}$ as above), for all $i\in A$, $w\in\Gamma^*$ and $x\in[0,1]$ we have

$$\begin{bmatrix}1-x & x\end{bmatrix}M(iw)=\begin{bmatrix}1-f_i(\mathcal{E}(w),x) & f_i(\mathcal{E}(w),x)\end{bmatrix}.$$

and for all $w'\notin A\Gamma^*$ we have

$$\begin{bmatrix}1-x & x\end{bmatrix}M(w')=\begin{bmatrix}1 & 0\end{bmatrix}.$$

(b9) Show that for any $w\in\Gamma^*$, $D_{\mathcal{E}(w)}\subseteq\{\mathcal{D}(v):v\in\Sigma'^*\}$.

(b10) By continuity in x and the uniform continuity in u of the $f_i(u,x)$ on $[0,1]^2$, show that if $(u_n)\in[0,1]^{\mathbb{N}}$ converges to some u^* then $\overline{\bigcup_{n=0}^{\infty}D_{u_n}}$ contains $\overline{D_{u^*}}$. *Hint: show that $D_{u^*}\subseteq\{x:\exists(s_n)_n$ such that $x=\lim_{n\to\infty}s_n$ and $s_n\in D_{u_n}$ for all $n\}$.*

(b11) Show that if $\frac{1}{4}$ is not an isolated cut-point of $\mathcal{E}$ then $\mathcal{D}$ has no isolated cut-points.

(b12) Let $u<\frac{1}{4}$ and D'_u be the smallest set such that $0\in D'_u$ and for all $x\in D'_u$, $i\in A$ and $u'\leqslant u$, $f_i(u',x)\in D'$. Show that D'_u is not dense in $[0,1]$. *Hint: proceed as in question **(b2)**.*

(b13) Show that if there exists $\varepsilon>0$ such that $\mathcal{E}(w)\leqslant\frac{1}{4}-\varepsilon$ for all $w\in\Sigma^*$, then $\mathcal{D}$ has at least one isolated cut-point. *Hint: show that $\{\mathcal{D}(w):w\in\Sigma'^*\}\subseteq D'_u$ for some $u<\frac{1}{4}$.*

(b14) Show that the following problem is undecidable: given a probabilistic automaton, decide whether it has at least one isolated cut-point. *Hint: consider applying the construction above to the automaton $\mathcal{E}=\mathcal{F}\cdot(1-\mathcal{F})$ for some arbitrary automaton $\mathcal{F}$.*

References

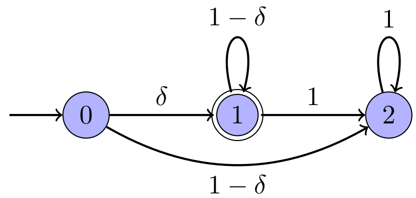
[Bal14] Kaspars Balodis. *Counting with Probabilistic and Ultrametric Finite Automata*, pages 3–16. Springer International Publishing, Cham, 2014.

[BMT77] Alberto Bertoni, Giancarlo Mauri, and Mauro Torelli. Some recursively unsolvable problems relating to isolated cutpoints in probabilistic automata. In *International Colloquium on Automata, Languages, and Programming*, pages 87–94. Springer, 1977.

Solutions to exercises

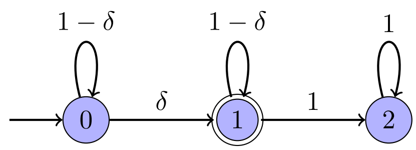
(a1) First observe that for any $i < j \leqslant n+1$, $a^i \not\equiv_L a_j$, since for $u = a^{n-i}$ (note that $i \leqslant n$ so $n - i \geqslant 0$), $a^i u \in C_n$ but $a^j u \notin L$. On the other hand, for any $i \geqslant n+1$, $a^{n+1} \equiv_L a^i$ since for all $u \in \Sigma^*$, $a^{n+1}u \notin L$ and $a^i u \notin L$. Hence, there are exactly $n+2$ equivalence classes and by Myhill-Nerode theorem, that's exactly the number of states of a minimal DFA recognizing C_n.

(a2) Consider the following automaton:



It is clear that the probability of a^ℓ being accepted is the probability of going from state 0 to state 1 (δ) and the probability of staying $\ell - 1$ times state 1 ($1 - \delta$).

(a3) Consider the following automaton:



An accepting run for a^ℓ first stay i times in state 0 then transitions to state 1 and stays $\ell - i - 1$ times in state 1. Therefore the probability is

$$\sum_{i=0}^{\ell-1}(1-\delta)^i\delta(1-\delta)^{n-1-i} = (1-\delta)^{\ell-1}\delta\sum_{i=0}^{\ell-1}1 = (1-\delta)^{\ell-1}\delta\ell.$$

(a4) Let $f(\ell) = (1-\delta)^{\ell-1}\delta\ell$, then $f'(\ell) = \log(1-\delta)(1-\delta)^{\ell-1}\delta\ell + (1-\delta)^{\ell-1}\delta = (1-\delta)^{\ell-1}(1+\ell\log(1-\delta))$. Since the derivate at 0 is positive and negative at the infinity, the maximum is attained once at ℓ such that $1 + \ell\log(1-\delta)$. Hence we choose δ such that the maximum is attained at $\ell = n$, that is $1 + n\log(1-\delta)$, so $\delta = 1 - e^{-1/n}$.

(a5) If we let $\delta = 1 - e^{-1/n}$, then $\mathcal{A}(a^\ell)$ is increasing from $\ell = 0$ to $\ell = n$ and decreasing from $\ell = n$ to infinity. Hence if we take $\lambda = \frac{1}{2}(\mathcal{A}(\ell^n) + \max(\mathcal{A}(a^{n-1}), \mathcal{A}(a^{n+1})))$ then $\mathcal{L}_\mathcal{A}(\lambda) = C_n$ and by construction λ is isolated. There is a striking comparison between DFAs and PFAs since the former requires n states to recognize C_n whereas the latter only requires 3 independently of n.

(b1) If $u = \frac{1}{4}$ then $0 \in D_u$ and for all $i \in \{0, 1, 2, 3\}$, $f_i(u, x) = \frac{1-u}{3}i + ux = \frac{i+x}{4} \in D_u$. In other words, D_u contains of 4−adic rationals which are clearly dense in $[0, 1]$.

(b2) Let $u < \frac{1}{4}$ and $\varepsilon = \frac{1}{4} - u > 0$. Let $X = [0, \frac{1}{4} - \varepsilon] \cup [\frac{1}{4}, 1]$. Observe that

$$f_0(u, X) = uX \subseteq [0, u] \subseteq [0, \tfrac{1}{4} - \varepsilon] \subseteq X$$

and for $i \in \{1, 2, 3\}$,

$$f_i(u, X) = \tfrac{1-u}{3}i + uX \subseteq \tfrac{1-u}{3}i + [0, u] \subseteq [\tfrac{1-u}{3}i, 1] \subseteq [\tfrac{1-u}{3}, 1] \subseteq [\tfrac{1}{4}, 1]$$

since $\frac{1-u}{3} \geqslant \frac{1}{4}$ since $u < \frac{1}{4}$. It follows that $0 \in X$ and $f_i(X) \subseteq X$, therefore $D_u \subseteq X$ since D_u is the smallest such set. Clearly X is not dense in $[0, 1]$ so D_u is not either.

(b3) Observe that by stochasticity, $S\mu(w) = \begin{bmatrix}1-x & x\end{bmatrix}$ for some x and $S\mu(w)T = x$. Hence, we are looking for matrices $\mu(i)$ such that

$$\begin{bmatrix}1-x & x\end{bmatrix}\mu(i) = \begin{bmatrix}1-f_i(u,x) & f_i(u,x)\end{bmatrix}.$$

By stochasticity, $\mu(i)$ is of the form

$$\mu(i) = \begin{bmatrix}1-a_i & a_i \\ 1-b_i & b_i\end{bmatrix}$$

and thus we must have $f_i(u, x) = \frac{1-u}{3}i + ux = a_i(1-x) + b_i x$. Hence we put $a_i = \frac{1-u}{3}i$ and $b_i = u + a_i$ which are indeed in $[0, 1]$.

(b4) By the previous question, if $u = \frac{1}{4}$ then $\{\mathcal{A}_u(w) : w \in A^*\} = D_u$ is dense in $[0,1]$ so it has no isolated cut-point. Conversely, if $u < \frac{1}{4}$ then D_u is not dense in $[0,1]$ so its complement contains an open interval I and therefore $\mathcal{A}_u$ has an isolated cut-point (the middle of the interval I).

(b5) $\mu'(c)$ is the product of two stochastic (non-rectangular) matrices. Let $w^{(1)}, \dots, w^{(k)} \in A^*$, then

$$\mathcal{D}(\sharp c^{(1)} \sharp c^{(2)} c \cdots \sharp c^{(k)} c) = S' \mu'(\sharp) \mu'(w^{(1)}) \mu'(\sharp) \cdots \mu'(\sharp) \mu'(w^{(k)}) \mu'(\sharp) = S' \begin{bmatrix} T_1^c & T_1 \\ T_2^c & T_2 \end{bmatrix} A_1 \cdots A_k \begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix} T'$$

where

$$\begin{aligned} A_i &= \begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix} \begin{bmatrix} \mu_1(w^{(i)}) & \mathbf{0} \\ \mathbf{0} & \mu_2(w^{(i)}) \end{bmatrix} \begin{bmatrix} T_1^c & T_1 \\ T_2^c & T_2 \end{bmatrix} \\ &= \begin{bmatrix} D_1 \mu_1(w^{(i)}) & \mathbf{0} \\ \mathbf{0} & D_2 \mu_2(w^{(i)}) \end{bmatrix} \begin{bmatrix} T_1^c & T_1 \\ T_2^c & T_2 \end{bmatrix} \\ &= \begin{bmatrix} D_1 \mu_1(w^{(i)}) T_1^c & D_1 \mu_1(w^{(i)}) T_1 \\ D_2 \mu_2(w^{(i)}) T_2^c & D_2 \mu_2(w^{(i)}) T_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \mathcal{B}(w^{(i)}) & \mathcal{B}(w^{(i)}) \\ 1 - \mathcal{C}(w^{(i)}) & \mathcal{C}(w^{(i)}) \end{bmatrix}. \end{aligned}$$

Furthermore,

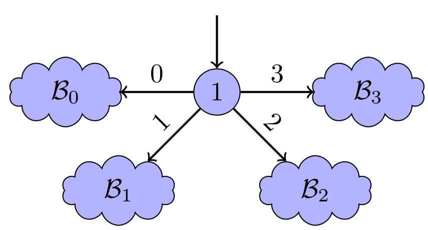
$$S' \begin{bmatrix} T_1^c & T_1 \\ T_2^c & T_2 \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} (T_1^c)^T & (T_2^c)^T \end{bmatrix} \begin{bmatrix} T_1^c & T_1 \\ T_2^c & T_2 \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} (T_1^c)^T T_1^c + (T_2^c)^T T_2^c & (T_1^c)^T T_1 + (T_2^c)^T T_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Indeed, if $v \in \{0,1\}^n$ then $v^T v^c = 0$ and $v^T v = v^T(\mathbf{1} - v^c) = v^T \mathbf{1} - v^T v^c = v^T \mathbf{1}$, therefore $(T_1^c)^T T_1^c + (T_2^c)^T T_2^c = (T_1^c)^T \mathbf{1} + (T_2^c)^T \mathbf{1} = \alpha$. Finally,

$$\begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix} T' = \begin{bmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} D_1 \mathbf{0} \\ D_2 \mathbf{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

by stochasticity of D_1 and D_2. This shows the result.

(b6) Consider the automaton $\mathcal{B}$ below:



It is sub-stochastic and can be made stochastic with a sink state. Clearly $\mathcal{B}(\varepsilon) = 0$ and $\mathcal{B}(iw) = \mathcal{B}_i(w)$ for all $w \in \Gamma^*$. Furthermore, if $w \notin \Gamma^*$ then $\mathcal{B}(iw) = 0$ since there $\mathcal{B}_i$ only has letters labelled by Γ (ie the transition will lead to the sink state). Finally, if the first letter is not in A, it will also lead to a sink state. The construction for $\mathcal{C}$ is exactly the same.

(b7) For each i, $1 - \mathcal{E}(w)$ corresponds to the complement and multiplying by $\frac{i}{3}$ can be done trivially by a convex combination of $1 - \mathcal{E}(w)$ and the automaton that has constant probability 0 for all words. Hence $\mathcal{B}_i$ is immediately seen to be a probabilistic automaton. Similarly, observe that $\mathcal{C}_i(w) = \frac{i}{3} + (1 - \frac{i}{3})\mathcal{E}(w)$ and hence is also a convex combination of $\mathcal{E}$ and the automaton that has constant probability 1.

(b8) By the previous questions, for all $i \in A$ and $w \in \Gamma^*$ we have ($\star$ denotes that the value is computed by stochasticity)

$$\begin{aligned} \begin{bmatrix} 1-x & x \end{bmatrix} M(iw) &= \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} 1 - \mathcal{B}(iw) & \mathcal{B}(iw) \\ 1 - \mathcal{C}(iw) & \mathcal{C}(iw) \end{bmatrix} \\ &= \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} 1 - \mathcal{B}_i(w) & \mathcal{B}_i(w) \\ 1 - \mathcal{C}_i(w) & \mathcal{C}_i(w) \end{bmatrix} \\ &= \begin{bmatrix} \star & (1-x)B_i(w) + C_i(w) \end{bmatrix} \\ &= \begin{bmatrix} \star & B_i(x) + (C_i(w) - B_i(w))x \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \star & \frac{1-\mathcal{E}(w)}{3}i + \mathcal{E}(w)x \end{bmatrix}$$
$$= \begin{bmatrix} \star & f_i(\mathcal{E}(w), x) \end{bmatrix}.$$

For any $w' \notin A\Gamma^*$, we have

$$\begin{aligned}
\begin{bmatrix} 1-x & x \end{bmatrix} M(w') &= \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} 1-\mathcal{B}(w') & \mathcal{B}(w') \\ 1-\mathcal{C}(w') & \mathcal{C}(w') \end{bmatrix} \\
&= \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}$$

(b9) We show a slightly stronger result. Let $X = \{\mathcal{D}(v) : v \in \sharp(\Sigma^*\sharp)^*\}$, then $0 \in X$ since $\mathcal{D}(\sharp) = 0$ by the calculations of question **(b5)**. Furthermore, if $x \in X$ then there exists $v \in \sharp(\Sigma^*\sharp)^*$ such that $S'\mu'(v) = \begin{bmatrix} 1-x & x \end{bmatrix}$. But then, by question **(b5)** and question **(b8)**, for every $i \in A$,

$$\mathcal{D}(viw\sharp) = S'\mu'(viw\sharp)T' = \begin{bmatrix} 1-x & x \end{bmatrix} M(iw) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-f_i(\mathcal{E}(w), x) & f_i(\mathcal{E}(w), x) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = f_i(\mathcal{E}(w), x).$$

It follows that $f_i(\mathcal{E}(w), x) \in X$. But $D_{\mathcal{E}(w)}$ is the smallest set that is stable under those operations, hence $D_{\mathcal{E}(w)} \subseteq X$.

(b10) Let $X = \{x : \exists (s_n)_n \text{ such that } x = \lim_{n\to\infty} s_n \text{ and } s_n \in D_{u_n} \text{ for all } n\}$. Clearly $X \subseteq Y := \overline{\bigcup_{n=0}^{\infty} D_{u_n}}$. Therefore if we show that $D_{u^*} \subseteq X$, we will have $D_{u^*} \subseteq Y$ and hence $\overline{D_{u^*}} \subseteq \overline{Y} = Y$ since Y is closed; which shows the result.

It remains to see that $D_{u^*} \subseteq X$: clearly $0 \in X$ since $0 \in D_{u_n}$ for all n. Let $x \in X$ and write $x = \lim_{n\to\infty} s_n$ where $s_n \in D_{u^*}$. Then that for any $i \in A$,

$$f_i(u^*, x) = f_i(u^*, \lim_n s_n) = \lim_n f_i(u^*, s_n)$$

by continuity of f_i. Furthermore, by the uniform continuity of f_i (continuity of the compact set $[0,1]^2$), there exists α such that for any n, $|f_i(u^*, s_n) - f_i(u_n, s_n)| \leqslant \alpha|u^* - u_n|$. Hence we can write $f_i(u^*, s_n) = f_i(u_n, s_n) + \varepsilon_n$ where $|\varepsilon_n| \leqslant \alpha|u^* - u_n| \to 0$ as $n \to \infty$. Now let $s'_n = f_i(u_n, s_n)$, then $s'_n \in D_{u_n}$ since $s_n \in D_{u_n}$ and $(s'_n)_n$ has a limit since $s'_n = f_i(u^*, s_n) - \varepsilon_n \to f_i(u^*, s)$ as shown above. Therefore,

$$f_i(u^*, x) = \lim_n f_i(u^*, s_n) = \lim_n s'_n + \varepsilon_n = \lim_s s'_n \in X.$$

This shows that X is stable under the application of the $f_i(u^*, \cdot)$, hence it contains D_{u^*} which is the smallest set to satisfy this condition.

(b11) If $\frac{1}{4}$ is not isolated then there exists a sequence $(w_n)_n$ of words such that $\mathcal{E}(w_n) \to \frac{1}{4}$. But then by the previous questions, we have

$$\bigcup_n D_{\mathcal{E}(w_n)} \subseteq \{\mathcal{D}(v) : v \in \Gamma'^*\}$$

and hence

$$\overline{D_{1/4}} \subseteq \overline{\bigcup_n D_{\mathcal{E}(w_n)}} \subseteq \overline{\{\mathcal{D}(v) : v \in \Gamma'^*\}}.$$

But $\overline{D_{1/4}} = [0,1]$ by question **(b1)**, hence $\overline{\{\mathcal{D}(v) : v \in \Gamma'^*\}} = [0,1]$ and therefore $\mathcal{D}$ cannot have an isolated cut-point.

(b12) The proof is essentially the same as in question **(b2)**. Let $\varepsilon = \frac{1}{4} - u > 0$ and $X = [0, \frac{1}{4} - \varepsilon] \cup [\frac{1}{4}, 1]$. Observe that for all $u' \leqslant u$,

$$f_0(u', X) = u'X \subseteq [0, u'] \subseteq [0, u] \subseteq [0, \tfrac{1}{4} - \varepsilon] \subseteq X$$

and for $i \in \{1, 2, 3\}$,

$$f_i(u', X) = \tfrac{1-u'}{3}i + u'X \subseteq \tfrac{1-u'}{3}i + [0, u'] \subseteq [\tfrac{1-u'}{3}i, 1] \subseteq [\tfrac{1-u'}{3}, 1] \subseteq [\tfrac{1}{4}, 1]$$

since $\frac{1-u'}{3} \geqslant \frac{1}{4}$ since $' \leqslant u < \frac{1}{4}$. It follows that $0 \in X$ and $f_i(X) \subseteq X$, therefore $D'_u \subseteq X$ since D'_u is the smallest such set. Clearly X is not dense in $[0,1]$ so D'_u is not either.

(b13) Let $X = \{\mathcal{D}(w) : w \in \Sigma'^*\}$. By questions **(b5)** and **(b8)**, we have that every $x \in X$ is either 0 or of the form $f_i(\mathcal{E}(v), x')$ for some $v \in \Gamma^*$ and $x' \in X$. But $\mathcal{E}(v) \leqslant \frac{1}{4} - \varepsilon$, hence by letting $u = \frac{1}{4} - \varepsilon < \frac{1}{4}$, we get that $X \subseteq X'_u$. It follows by question **(b12)** that X is not dense in $[0,1]$. Consequently, there is an open interval I in $[0,1]$ that does not intersect X and the center of this interval is an isolated cut-point of $\mathcal{D}$.

(b14) We show that the problem is undecidable by reducing from the problem of deciding whether $\frac{1}{2}$ is an isolated cut-point of a given automaton. The will show the result since the latter is an undecidable problem.

Let $\mathcal{F}$ be any automaton on alphabet Γ. We can build $\mathcal{E}$ such that $\mathcal{E}(w) = \mathcal{F}(w) \cdot (1 - \mathcal{F}(w))$ for all $w \in \Gamma^*$ by the product construction. Then observe that $\mathcal{E}(w) \leqslant \frac{1}{4}$ for all w. Furthermore, $\frac{1}{4}$ is isolated for $\mathcal{E}$ if and only if $\frac{1}{2}$ is isolated for $\mathcal{F}$. We now build $\mathcal{D}$ as done in the questions above. By questions **(b11)** and **(b13)**, we have that $\mathcal{D}$ has an isolated cut-point if and only if $\frac{1}{4}$ is isolated for $\mathcal{E}$. Therefore we have reduced the problem of whether $\frac{1}{2}$ is isolated for $\mathcal{F}$ to the problem of deciding whether $\mathcal{D}$ has an isolated cut-point.