On generable functions and a universal ordinary differential equation

Amaury Pouly

1Max Planck Institute for Software Systems, Germany

26 july 2017
Polynomial Differential Equations

\[ k \quad k \quad u \quad \times \quad uv \]

\[ u \quad + \quad u+v \quad u \quad \int \quad \int u \]

General Purpose Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks:
- chemical
- enzymatic

polynomial differential equations:
\[
\begin{cases}
y(0) = y_0 \\
y'(t) = p(y(t))
\end{cases}
\]

- Rich class
- Stable (+, ×, ○, /, ED)
- No closed-form solution
Example of differential equation

\[ \ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \]

General Purpose Analog Computer (GPAC)

Shannon’s model of the Differential Analyser

\[
\begin{align*}
  y_1' &= y_2 \\
  y_2' &= -\frac{g}{\ell} y_3 \\
  y_3' &= y_2 y_4 \\
  y_4' &= -y_2 y_3
\end{align*}
\]

\[
\begin{align*}
  y_1 &= \theta \\
  y_2 &= \dot{\theta} \\
  y_3 &= \sin(\theta) \\
  y_4 &= \cos(\theta)
\end{align*}
\]
Some motivation

Polynomial ODEs correspond to analog computers:

Differential Analyser

British Navy mecanical computer
Some motivation

Polynomial ODEs correspond to analog computers:

- They are equivalent to Turing machines!
- One can characterize $P$ with pODEs
- There exists a universal pODE for continuous functions

Take away: polynomial ODEs are a natural programming language.
Outline

1. The theory of generable functions
2. A universal differential equation
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d$, $y : \mathbb{R} \to \mathbb{R}^d$

**Note**: existence and unicity of $y$ by Cauchy-Lipschitz theorem.
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Example**

\( f(x) = x \) \quad ➤ identity

\[
y(0) = 0, \quad y' = 1 \quad \leadsto \quad y(x) = x
\]

**Types**

- \( d \in \mathbb{N} \) : dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \) : field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] \) : polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d \)
Generable functions (total, univariate)

<table>
<thead>
<tr>
<th>Definition</th>
<th>Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f : \mathbb{R} \to \mathbb{R} ) is generable if there exists ( d, p ) and ( y_0 ) such that the solution ( y ) to ( y(0) = y_0, \quad y'(x) = p(y(x)) ) satisfies ( f(x) = y_1(x) ) for all ( x \in \mathbb{R} ).</td>
<td>( d \in \mathbb{N} ): dimension ( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} ): field ( p \in \mathbb{K}^d[\mathbb{R}^n] ): polynomial vector (coef. in ( \mathbb{K} )) ( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d )</td>
</tr>
</tbody>
</table>

Example: \( f(x) = x^2 \) ▶ squaring

\[
\begin{align*}
y_1(0) &= 0, \quad y_1' = 2y_2 \quad \leadsto \quad y_1(x) = x^2 \\
y_2(0) &= 0, \quad y_2' = 1 \quad \leadsto \quad y_2(x) = x
\end{align*}
\]
Generable functions (total, univariate)

Definition

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

Types

- \( d \in \mathbb{N} : \) dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} : \) field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] : \) polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

Example: \( f(x) = x^n \)  \( \rightarrow \) \( n^{th} \) power

\[
\begin{align*}
y_1(0) &= 0, & y_1' &= ny_2 \\
y_2(0) &= 0, & y_2' &= (n - 1)y_3 \\
\vdots & & \vdots \\
y_n(0) &= 0, & y_n &= 1 \\
\end{align*}
\]

\( \leadsto \)

\[
\begin{align*}
y_1(x) &= x^n \\
y_2(x) &= x^{n-1} \\
\vdots & & \vdots \\
y_n(x) &= x \\
\end{align*}
\]
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$

**Example**: $f(x) = \exp(x)$  

$\triangleright$ exponential

$$y(0) = 1, \quad y' = y \quad \Rightarrow \quad y(x) = \exp(x)$$
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \) : dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \) : field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] \) : polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example** : \( f(x) = \sin(x) \) or \( f(x) = \cos(x) \)

\[
\begin{align*}
y_1(0) &= 0, \quad y_1' = y_2 \quad \sim \quad y_1(x) = \sin(x) \\
y_2(0) &= 1, \quad y_2' = -y_1 \quad \sim \quad y_2(x) = \cos(x)
\end{align*}
\]
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$

**Example:** $f(x) = \tanh(x)$  

\[ y(0) = 0, \quad y' = 1 - y^2 \quad \mapsto \quad y(x) = \tanh(x) \]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} : \) dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} : \) field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] : \) polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example**

\( f(x) = \frac{1}{1+x^2} \) ➤ rational function

\[
f'(x) = \frac{-2x}{(1+x^2)^2} = -2xf(x)^2
\]

\[
y_1(0)= 1, \quad y'_1 = -2y_2y_1^2 \quad \leadsto \quad y_1(x) = \frac{1}{1+x^2}
\]

\[
y_2(0)= 0, \quad y'_2 = 1 \quad \leadsto \quad y_2(x) = x
\]
Generable functions (total, univariate)

**Definition**

\[ f : \mathbb{R} \rightarrow \mathbb{R} \] is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]
satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \): dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \): field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] \): polynomial
  vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d \), \( y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example:** \( f = g \pm h \)  

\[
(f \pm g)' = f' \pm g'
\]

*assume:*

\[
\begin{align*}
z(0) &= z_0, & z' &= p(z) & \leadsto & z_1 &= g \\
w(0) &= w_0, & w' &= q(w) & \leadsto & w_1 &= h
\end{align*}
\]

*then:*

\[
\begin{align*}
y(0) &= z_{0,1} + w_{0,1}, & y' &= p_1(z) \pm q_1(w) & \leadsto & y &= z_1 \pm w_1
\end{align*}
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} : \) dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} : \) field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] : \) polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d \)

**Example:** \( f = gh \)  ► product

\[
(gh)' = g'h + gh'
\]

**assume:**

\[
\begin{align*}
z(0) &= z_0, & z' &= p(z) \\
w(0) &= w_0, & w' &= q(w)
\end{align*}
\]

**then:**

\[
\begin{align*}
y(0) &= z_{0,1}w_{0,1}, & y' &= p_1(z)w_1 + z_1q_1(w) \quad \sim \quad y = z_1w_1
\end{align*}
\]
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is generable if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

**Example**

$f = \frac{1}{g} \quad \triangleright$ inverse

$$f' = \frac{-g'}{g^2} = -g'f^2$$

**assume**:

$$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$$

**then**:

$$y(0) = \frac{1}{z_{0,1}}, \quad y' = -p_1(z)y^2 \quad \leadsto \quad y = \frac{1}{z_1}$$
**Generable functions (total, univariate)**

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is *generable* if there exists $d, p$ and $y_0$ such that the solution $y$ to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example:** $f = \int g$  ⇒ integral

\[\text{assume:} \quad z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g\]

\[\text{then:} \quad y(0) = 0, \quad y' = z_1 \quad \leadsto \quad y = \int z_1\]

**Types**

- $d \in \mathbb{N}$ : dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$ : field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is generable if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example**: $f = g'$

\[
\begin{align*}
  f' &= g'' = (p_1(z))' = \nabla p_1(z) \cdot z' \\
\end{align*}
\]

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

*assume*: 

\[
\begin{align*}
  z(0) &= z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g \\
\end{align*}
\]

*then*: 

\[
\begin{align*}
  y(0) &= p_1(z_0), \quad y' = \nabla p_1(z) \cdot p(z) \quad \leadsto \quad y = z_1'' \\
\end{align*}
\]
### Generable functions (total, univariate)

**Definition**

A function \( f : \mathbb{R} \to \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \): dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \): field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] \): polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d \)

**Example:** \( f = g \circ h \) → composition

\[
(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'
\]

**Assume:**

- \( z(0) = z_0, \quad z' = p(z) \sim z_1 = g \)
- \( w(0) = w_0, \quad w' = q(w) \sim w_1 = h \)

**Then:**

- \( y(0) = z(w_0), \quad y' = p(y)z_1 \sim y = z \circ h \)
Generable functions (total, univariate)

<table>
<thead>
<tr>
<th>Definition</th>
<th>Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f : \mathbb{R} \rightarrow \mathbb{R}$ is generable if there exists $d, p$ and $y_0$ such that the solution $y$ to $y(0) = y_0$, $y'(x) = p(y(x))$ satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.</td>
<td>$d \in \mathbb{N}$ : dimension</td>
</tr>
<tr>
<td>$\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$ : field</td>
<td></td>
</tr>
<tr>
<td>$p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)</td>
<td></td>
</tr>
<tr>
<td>$y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$</td>
<td></td>
</tr>
</tbody>
</table>

Example : $f = g \circ h$  ► composition

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

assume :

$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$

$w(0) = w_0, \quad w' = q(w) \quad \leadsto \quad w_1 = h$

then :

$y(0) = z(w_0), \quad y' = p(y)z_1 \quad \leadsto \quad y = z \circ h$

Is this coefficient in $\mathbb{K}$?
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example:** $f = g \circ h$  

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

Is this coefficient in $\mathbb{K}$? Fields with this property are called **generable**.
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} : \) dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} : \) field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] : \) polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example** :

\( f' = \tanh \circ f \)  

\( f'' = (\tanh' \circ f)f' = (1 - (\tanh \circ f)^2)f' \)

\[
\begin{align*}
y_1(0) &= f(0), & y_1' &= y_2 \\
y_2(0) &= \tanh(f(0)), & y_2' &= (1 - y_2^2)y_2
\end{align*}
\]

\( \mapsto \)

\[
\begin{align*}
y_1(x) &= f(x) \\
y_2(x) &= \tanh(f(x))
\end{align*}
\]
**Generable functions (total, univariate)**

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example** : $f(0) = f_0$, $f' = g \circ f$

**Types**

- $d \in \mathbb{N}$ : dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$ : field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d$, $y : \mathbb{R} \rightarrow \mathbb{R}^d$

**Initial Value Problem (IVP)**

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

**assume** :

$$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$$

**then** :

$$y(0) = p_1(z_0), \quad y' = \nabla p_1(z) \cdot p(z) \quad \leadsto \quad y = z_1''$$
Generable functions: a first summary

Nice theory for the class of total and univariate generable functions:
- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
Generable functions: a first summary

Nice theory for the class of total and univariate generable functions:
- analytic
- contains polynomials, \( \sin, \cos, \tanh, \exp \)
- stable under \( \pm, \times, /, \circ \) and Initial Value Problems (IVP)
- technicality on the field \( \mathbb{K} \) of coefficients for stability under \( \circ \)

Limitations:
- total functions
- univariate
Generable functions (generalization)

**Definition**

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **generable** if $X$ is open connected and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \quad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

$J_y(x) =$ Jacobian matrix of $y$ at $x$

**Notes:**

- Partial differential equation!
- Unicity of solution $y$...
- **but not existence** (ie you have to show it exists)

**Types**

- $n \in \mathbb{N} :$ input dimension
- $d \in \mathbb{N} :$ dimension
- $p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] :$ polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d, y : X \to \mathbb{R}^d$
Generable functions (generalization)

**Definition**

\[ f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \] is generable if \( X \) is open connected and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) = \) Jacobian matrix of \( y \) at \( x \)

**Example**:

\[ f(x_1, x_2) = x_1x_2^2 \quad (n = 2, d = 3) \]

\[
y(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[ \sim \quad y(x) = \begin{pmatrix} x_1x_2^2 \\ x_1 \\ x_2 \end{pmatrix} \]

**Types**

- \( n \in \mathbb{N} \) : input dimension
- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] \) : polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)
Generable functions (generalization)

**Definition**

\[ f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \text{ is generable if } X \text{ is open connected and } \exists d, p, x_0, y_0, y \text{ such that} \]

\[ y(x_0) = y_0, \quad J_y(x) = p(y(x)) \]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) = \) Jacobian matrix of \( y \) at \( x \)

**Types**

- \( n \in \mathbb{N} : \) input dimension
- \( d \in \mathbb{N} : \) dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] : \) polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)

**Example:**

\[ f(x_1, x_2) = x_1 x_2^2 \quad \text{► monomial} \]

- \( y_1(0, 0) = 0, \quad \partial_{x_1} y_1 = y_3^2, \quad \partial_{x_2} y_1 = 3y_2 y_3 \quad \leadsto \quad y_1(x) = x_1 x_2^2 \)
- \( y_2(0, 0) = 0, \quad \partial_{x_1} y_2 = 1, \quad \partial_{x_2} y_2 = 0 \quad \leadsto \quad y_2(x) = x_1 \)
- \( y_3(0, 0) = 0, \quad \partial_{x_1} y_3 = 0, \quad \partial_{x_2} y_3 = 1 \quad \leadsto \quad y_3(x) = x_2 \)

This is tedious!
Generable functions (generalization)

Definition

\( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is generable if \( X \) is open connected and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) \) = Jacobian matrix of \( y \) at \( x \)

Last example: \( f(x) = \frac{1}{x} \) for \( x \in (0, \infty) \)

\( y(1) = 1, \quad \partial_x y = -y^2 \quad \sim \quad y(x) = \frac{1}{x} \)

Types

- \( n \in \mathbb{N} : \) input dimension
- \( d \in \mathbb{N} : \) dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] : \) polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)

▶ inverse function
Nice theory for the class of multivariate generable functions (over connected domains):

- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains):

- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $K$ of coefficients for stability under $\circ$

Natural questions:

- analytic $\rightarrow$ isn’t that very limited?
- can we generable all analytic functions?
Generable functions: summary

Nice theory for the class of multivariate generable functions (over connected domains):

- analytic
- contains polynomials, \( \sin, \cos, \tanh, \exp \)
- stable under \( \pm, \times, /, \circ \) and Initial Value Problems (IVP)
- technicality on the field \( \mathbb{K} \) of coefficients for stability under \( \circ \)

Natural questions:

- analytic \( \rightarrow \) isn’t that very limited?
- can we generable all analytic functions? No

Riemann \( \Gamma \) and \( \zeta \) are not generable.
Computing with generable functions

Generable functions

\[ y(0) = y_0 \quad y' = p(y) \]

\[ f(x) = y_1(x) \quad x \in \mathbb{R} \]

\[ \sin, \cos, \exp, \log, \ldots \subseteq \text{Analytic} \]
Computing with generable functions

Generable functions

\[ y(0) = y_0 \quad y' = p(y) \]

\[ f(x) = y_1(x) \quad x \in \mathbb{R} \]

\[ \sin, \cos, \exp, \log, \ldots \subseteq \text{Analytic} \]

Analog computable function

\[ y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \]

\[ |f(x) - y_1(t)| \leq 2^{-t} \]
Computing with generable functions

Generable functions

\[ y(0) = y_0 \quad y' = p(y) \]

\[ f(x) = y_1(x) \quad x \in \mathbb{R} \]

\[ \sin, \cos, \exp, \log, \ldots \subset \text{Analytic} \]

Analog computable function

\[ y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \]

\[ |f(x) - y_1(t)| \leq 2^{-t} \]

Turing powerful

Theorem (Bournez et al., 2007)

\[ f : [a, b] \to \mathbb{R} \text{ is computable}^a \text{ iff } f \text{ is analog computable.} \]

\[ \text{a. In the sense of Computable Analysis.} \]
Computing with generable functions

Generable functions

\[ y(0) = y_0 \quad y' = p(y) \]
\[ f(x) = y_1(x) \quad x \in \mathbb{R} \]

sin, cos, exp, log, … \subset \text{ Analytic} \]

Analog computable function

\[ y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \]
\[ |f(x) - y_1(t)| \leq 2^{-t} \]

Turing powerful

Question: reformulate analog computability with generable functions?
Computing with generable functions

Generable functions

\[ y(0) = y_0 \quad y' = p(y) \]

\[ f(x) = y_1(x) \quad x \in \mathbb{R} \]

\[ \sin, \cos, \exp, \log, \ldots \subseteq \text{Analytic} \]

Analog computable function

\[ y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \]

\[ |f(x) - y_1(t)| \leq 2^{-t} \]

Turing powerful

Theorem

\( f : [a, b] \rightarrow \mathbb{R} \) is computable\(^a\) iff \( \exists \) a generable function \( g \) such that

\[ |f(x) - g(x, t)| \leq 2^{-t} \]

for all \( x \in [a, b] \) and \( t \geq 0 \).

\(^a\) In the sense of Computable Analysis.
It is all about the coefficients

Theorem

\[ f : [a, b] \rightarrow \mathbb{R} \text{ is computable iff } \exists \text{ a generable function } g \text{ such that} \]

\[ |f(x) - g(x, t)| \leq 2^{-t} \quad \text{for all } x \in [a, b] \text{ and } t \geq 0. \]

Which coefficients are used?
It is all about the coefficients

**Theorem**

\( f : [a, b] \rightarrow \mathbb{R} \) is computable iff \( \exists \) a *generable* function \( g \) such that

\[
|f(x) - g(x, t)| \leq 2^{-t} \quad \text{for all } x \in [a, b] \text{ and } t \geq 0.
\]

Which coefficients are used? We need to talk about \( \mathbb{K} \)...

- original proof: unclear, something like \( \mathbb{Q}(\pi, e, \text{others?}) \)
It is all about the coefficients

**Theorem**

\( f : [a, b] \rightarrow \mathbb{R} \) is computable iff \( \exists \) a **generable** function \( g \) such that

\[
|f(x) - g(x, t)| \leq 2^{-t}
\]

for all \( x \in [a, b] \) and \( t \geq 0 \).

**Which coefficients are used?** We need to talk about \( \mathbb{K} \)...  
- original proof: unclear, something like \( \mathbb{Q}(\pi, e, \text{others?}) \)
- more recent proof: \( \mathbb{R}_G = \) smallest **generable** field

**Theorem**

\( \mathbb{Q} \subsetneq \mathbb{R}_G \subsetneq \mathbb{R}_P = \) polytime reals.
It is all about the coefficients

**Theorem**

\[ f : [a, b] \rightarrow \mathbb{R} \text{ is computable iff } \exists \text{ a generable function } g \text{ such that} \]

\[ |f(x) - g(x, t)| \leq 2^{-t} \quad \text{for all } x \in [a, b] \text{ and } t \geq 0. \]

Which coefficients are used? We need to talk about \( \mathbb{K} \... \)

- original proof: unclear, something like \( \mathbb{Q}(\pi, e, \text{others?}) \)
- more recent proof: \( \mathbb{R}_G = \text{smallest generable field} \)

**Theorem**

\( \mathbb{Q} \subset \mathbb{R}_G \subset \mathbb{R}_P = \text{polytime reals.} \)

- this year: \( \mathbb{Q} \) is enough! (and we can even characterize polytime)
It is all about the coefficients

Theorem

\[ f : [a, b] \rightarrow \mathbb{R} \text{ is computable iff } \exists \text{ a generable function } g \text{ such that} \]
\[ |f(x) - g(x, t)| \leq 2^{-t} \quad \text{for all } x \in [a, b] \text{ and } t \geq 0. \]

Which coefficients are used? We need to talk about \( \mathbb{K} \ldots \)

- original proof: unclear, something like \( \mathbb{Q}(\pi, e, \text{others?}) \)
- more recent proof: \( \mathbb{R}_G = \text{smallest generable field} \)

Theorem

\( \mathbb{Q} \subsetneq \mathbb{R}_G \subseteq \mathbb{R}_P = \text{polytime reals.} \)

- this year: \( \mathbb{Q} \) is enough! (and we can even characterize polytime)

What happens if we take \( \mathbb{K} = \mathbb{R} ? \)
1. The theory of generable functions
2. A universal differential equation
Theorem (Rubel, 1981)
For any $f \in C^0(\mathbb{R})$ and $\epsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$3y'^4y''y''''^2 - 4y'^4y'''^2y'''' + 6y'^3y''^2y''''y''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y'''^3y''^2 + 12y''''^7 = 0$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \epsilon(t).$$
Theorem (Rubel, 1981)

There exists a fixed $k$ and nontrivial polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$p(y, y', \ldots, y^{(k)}) = 0$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Universal differential algebraic equation (Rubel)

Open Problem (Rubel)
Can we have unicity of the solution with initial conditions?

Theorem (Rubel, 1981)
There exists a fixed $k$ and nontrivial polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_0)$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$p(y, y', \ldots, y^{(k)}) = 0$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Rubel’s ("disappointing") proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

  It satisfies $\left(1 - t^2\right)^2 f''(t) + 2tf'(t) = 0$. 

[Graph showing a function with a peak and trough]
Rubel’s ("disappointing") proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.
  It satisfies $(1 - t^2)^2 f''(t) + 2tf'(t) = 0$.
- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at + b)$ satisfies
  
  $$3y'^4 y'' y''''^2 - 4y'^4 y''^2 y'''' + 6y'^3 y'' y''' y'''' + 24y'^2 y'' y'''^4 y'''' + 12y'^7 = 0$$

Translation and rescaling:
Rubel’s ("disappointing") proof in one slide

- Take \( f(t) = e^{\frac{-1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise.
  
  It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0\).

- For any \( a, b, c \in \mathbb{R} \), \( y(t) = cf(at + b) \) satisfies
  
  \[
  3y^4y''y''''2 - 4y^4y''y'''2 + 6y^3y''y''''3 + 24y^2y'''y''''3 - 12y''y''''y'''3 - 29y''y'''y''''2 + 12y''''7 = 0
  \]

- Can glue together arbitrary many such pieces
Rubel’s ("disappointing") proof in one slide

- Take \( f(t) = e^{\frac{-1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise.
  
  It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0\).

- For any \( a, b, c \in \mathbb{R} \), \( y(t) = cf(at + b) \) satisfies

\[
3y^4 y'' y''''^2 - 4y^4 y''^2 y'''' + 6y^3 y''^2 y'''' + 24y^2 y''^4 y'''' - 12y^3 y'' y'''^3 - 29y^2 y''' y''''^2 + 12y''^7 = 0
\]

- Can glue together arbitrary many such pieces
- Can arrange so that \( \int f \) is solution: piecewise pseudo-linear
Rubel’s ("disappointing") proof in one slide

- Take \( f(t) = e^{1-t^2} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise.
  It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0\).
- For any \( a, b, c \in \mathbb{R} \), \( y(t) = cf(at + b) \) satisfies
  \[3y''^4 y''''^2 - 4y''^4 y''^2 y'''' + 6y'''^3 y''^2 y''' + 24y''^4 y''''' - 12y'''^3 y'' y'''^3 - 29y''^2 y'''^3 y''''^2 + 12y''^7 = 0\]
- Can glue together arbitrary many such pieces
- Can arrange so that \( \int f \) is solution: piecewise pseudo-linear

Conclusion: Rubel’s equation allows any piecewise pseudo-linear functions, and those are dense in \( C^0 \)
The problem with Rubel’s DAE

The solution $y$ is not unique, **even with added initial conditions**:

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \quad y'(0) = \alpha_1, \ldots, \quad y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel’s proof to work!
The problem with Rubel’s DAE

The solution \( y \) is not unique, \textbf{even with added initial conditions}:

\[
p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, \ y^{(k)}(0) = \alpha_k
\]

In fact, this is fundamental for Rubel’s proof to work!

- Rubel’s statement: this DAE is universal
- More realistic interpretation: this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE \( y' = p(y) \)?

\textbf{Note}: explicit polynomial ODE \( \Rightarrow \) unique solution
Universal explicit ordinary differential equation

Theorem (universal pIVP)
There exists a fixed (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a unique solution $y : \mathbb{R} \rightarrow \mathbb{R}^d$ and $\forall t \in \mathbb{R},$

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$
Universal explicit ordinary differential equation

Notes:
- **system** of ODEs,
- $y$ must be analytic,
- we need $d \approx 300$.

**Theorem (universal pIVP)**

There exists a fixed (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a unique solution $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$
Theorem (universal generable function)

There exists a fixed generable function $g : \subseteq \mathbb{R}^{d+1} \to \mathbb{R}$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$|f(t) - g(\alpha, t)| \leq \varepsilon(t) \quad \forall t \in \mathbb{R}.$$ 

Note: $\alpha$ is usually transcendental, and typically not in $\mathbb{R}_G$...
Corollary of main result

There exists a **fixed** $k$ and nontrivial polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, \ y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** $y : \mathbb{R} \rightarrow \mathbb{R}$ and $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Proof by picture (simplified)

binary stream generator : digits of $\alpha \in \mathbb{R}$

$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2} \tanh(\mu \sin(2\alpha \pi 4^{\text{round}(t-1/4, \lambda)} + 4\pi/3))$$

It’s horribly generable

round is a mysterious rounding function...
Proof by picture (simplified)

binary stream generator: digits of $\alpha \in \mathbb{R}$

$$
\begin{array}{c}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
$$

dyadic stream generator:

$$
d_i = m_i 2^{-d_i}, \quad a_i = 9i + \sum_{j<i} d_j
$$

$$
f(\alpha, \gamma, t) = \sin(2\alpha \pi 2^{\text{round}(t-1/4, \gamma)})
$$

round is a mysterious rounding function...
Proof by picture (simplified)
Proof by picture (simplified)
Proof by picture (simplified)
Proof by picture (simplified)
Proof by picture (simplified)
This copy operation is the “non-trivial” part.
We can do almost piecewise constant functions...
We can do **almost piecewise constant functions**...
- ...that are **bounded by 1**...
- ...and have **super slow changing frequency**.
We can do **almost piecewise constant functions**...  
  - ...that are **bounded by 1**...  
  - ...and have **super slow changing frequency**.

How do we go to arbitrarily large and growing functions? **Can a polynomial ODE even have arbitrary growth?**
Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \quad \sim \quad y_1(t) = \exp(t)$$
Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[
\begin{align*}
y_1' &= y_1 \\
y_2' &= y_1 y_2
\end{align*}
\implies
\begin{align*}
y_1(t) &= \exp(t) \\
y_2(t) &= \exp(\exp(t))
\end{align*}
\]
An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$
\begin{align*}
  y_1' &= y_1 & \Rightarrow & & y_1(t) &= \exp(t) \\
  y_2' &= y_1 y_2 & \Rightarrow & & y_2(t) &= \exp(\exp(t)) \\
  & \cdots & \cdots & \cdots \\
  y_n' &= y_1 \cdots y_n & \Rightarrow & & y_n(t) &= \exp(\cdots \exp(t) \cdots ) := e_n(t)
\end{align*}
$$
An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[ y_1' = y_1 \sim y_1(t) = \exp(t) \]
\[ y_2' = y_1 y_2 \sim y_2(t) = \exp(\exp(t)) \]
\[ \vdots \]
\[ y_n' = y_1 \cdots y_n \sim y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t) \]

Conjecture (Emile Borel, 1899)

With $n$ variables, cannot do better than $O(t(e_n(At^k)))$. 

Counter-example (Vijayaraghavan, 1932)

\[ \frac{1}{2} - \cos(t) - \cos(\alpha t) \]
An old question on growth

$$e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions})$$

Conjecture (Emile Borel, 1899)

With $n$ variables, cannot do better than $O(t(e_n(At^k)))$.

Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$

Sequence of arbitrarily growing spikes.
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

Conjecture (Emile Borel, 1899)

With \( n \) variables, cannot do better than \( O(t^n(A)^k) \).

Counter-example (Vijayaraghavan, 1932)

\[
\frac{1}{2 - \cos(t) - \cos(\alpha t)}
\]

Sequence of arbitrarily growing spikes. But not good enough for us.
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

Conjecture (Emile Borel, 1899)

With \( n \) variables, cannot do better than \( O_t(e_n(At^k)) \).

Counter-example (Vijayaraghavan, 1932)

\[
\frac{1}{2 - \cos(t) - \cos(\alpha t)}
\]

Theorem (In the paper)

There exists a polynomial \( p : \mathbb{R}^d \to \mathbb{R}^d \) such that for any continuous function \( f : \mathbb{R}_{\geq 0} \to \mathbb{R} \), we can find \( \alpha \in \mathbb{R}^d \) such that

\[ y(0) = \alpha, \quad y'(t) = p(y(t)) \]

satisfies

\[ y_1(t) \geq f(t), \quad \forall t \geq 0. \]
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

**Conjecture (Emile Borel, 1899)**

With \( n \) variables, cannot do better than \( O(t e_n(A t^k)) \).

**Counter-example (Vijayaraghavan, 1932)**

\[
\frac{1}{2 - \cos(t) - \cos(\alpha t)}
\]

**Theorem (In the paper)**

There exists a polynomial \( p : \mathbb{R}^d \to \mathbb{R}^d \) such that for any continuous function \( f : \mathbb{R}_{\geq 0} \to \mathbb{R} \), we can find \( \alpha \in \mathbb{R}^d \) such that

\[
y(0) = \alpha, \quad y'(t) = p(y(t))
\]

satisfies

\[
y_1(t) \geq f(t), \quad \forall t \geq 0.
\]

**Note:** both results require \( \alpha \) to be **transcendental**. Conjecture still open for **rational** coefficients.
Goal

Iterate $f$ with a GPAC: $y(n) \approx f^n([x])$
Goal

Iterate \( f \) with a GPAC: \( y(n) \approx f^{[n]}([x]) \)
Goal

Iterate $f$ with a GPAC: $y(n) \approx f[n](x)$

\[
y'(\approx 0) \quad \frac{1}{2} \quad y' \approx z - y \quad 1 \quad \frac{3}{2} \quad 2
\]

\[
z' \approx f(y) - z \quad z' \approx 0
\]
Goal
Iterate $f$ with a GPAC: $y(n) \approx f^{[n]}(x)$
A computability question

Theorem (universal pIVP)
There exists a **fixed** (vector of) polynomial \( p \) such that for any \( f \in C^0(\mathbb{R}) \) and \( \varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0}) \), there exists \( \alpha \in \mathbb{R}^d \) such that

\[
y(0) = \alpha, \quad y'(t) = p(y(t))
\]

has a **unique solution** \( y : \mathbb{R} \rightarrow \mathbb{R}^d \) and \( \forall t \in \mathbb{R}, \)

\[
|y_1(t) - f(t)| \leq \varepsilon(t).
\]

Theorem (universal generable function)
There exists a **fixed** generable function \( g : \subseteq \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) such that for any \( f \in C^0(\mathbb{R}) \) and \( \varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0}) \), there exists \( \alpha \in \mathbb{R}^d \) such that

\[
|f(t) - g(\alpha, t)| \leq \varepsilon(t) \quad \forall t \in \mathbb{R}.
\]

**Question**: is \( \alpha \) computable from \( f \) and \( \varepsilon \)?
A computability question

Theorem (universal pIVP)
There exists a fixed (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

\[ y(0) = \alpha, \quad y'(t) = p(y(t)) \]

has a unique solution $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

\[ |y_1(t) - f(t)| \leq \varepsilon(t). \]

Theorem (universal generable function)
There exists a fixed generable function $g : \subseteq \mathbb{R}^{d+1} \to \mathbb{R}$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

\[ |f(t) - g(\alpha, t)| \leq \varepsilon(t) \quad \forall t \in \mathbb{R}. \]

Claim (WIP) : $(f, \varepsilon) \mapsto \alpha$ is computable (for some reasonable representation)
Conclusion

This talk
theory of generable functions
positive answer to Rubel’s open problem

Take home
programming with polynomial ODE is nice and fun

Possible development

Each universal ODE defines a map :

\((f, \varepsilon) \in C^0 \times C^0 \mapsto \alpha \in \mathbb{R}\)

Kolmogorov-like complexity for continuous functions?
Digital vs analog computers
Digital vs analog computers

VS
Church Thesis

All reasonable models of computation are equivalent.

Computability

- Turing machine
- boolean circuits
- lambda calculus
- recursive functions
- logic
- quantum
- analog

discrete
continuous
Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Universal differential equations

Generable functions

subclass of analytic functions

any computable function

Computable functions
A new notion of computability

Almost-Theorem

\( f : [0, 1] \to \mathbb{R} \) is \textbf{computable} if and only if there exists \( \tau > 1 \), \( y_0 \in \mathbb{R}^d \) and \( p \) polynomial such that

\[
y'(0) = y_0, \quad y'(t) = p(y(t))
\]

satisfies

\[
|f(x) - y(x + n\tau)| \leq 2^{-n}, \quad \forall x \in [0, 1], \forall n \in \mathbb{N}
\]