On generable functions and a universal ordinary differential equation

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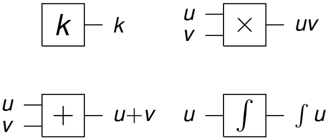
Polynomial Differential Equations



General Purpose
Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks :

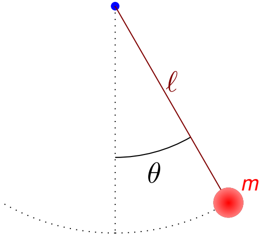
- chemical
- enzymatic

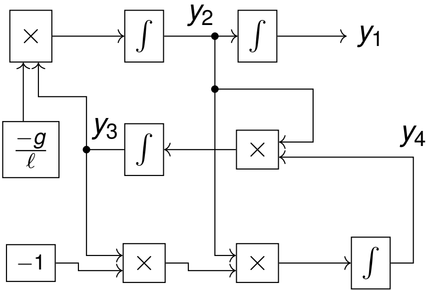
polynomial differential equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable ($+, \times, \circ, /$, ED)
- No closed-form solution

Example of differential equation





General Purpose Analog Computer (GPAC)
Shannon's model of the Differential Analyser

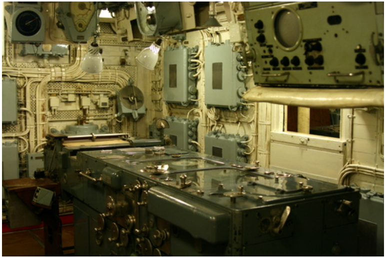
$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{\ell} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

Some motivation

Polynomial ODEs correspond to analog computers :

Differential Analyser



British Navy mecanical computer

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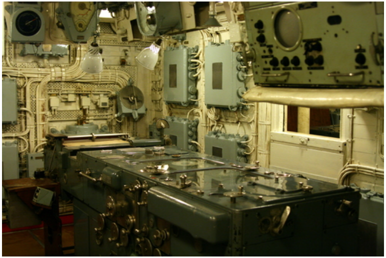

British Navy mecanical computer

- They are **equivalent** to Turing machines !
- One can **characterize P** with pODEs
- There exists a **universal pODE** for continuous functions

Take away : polynomial ODEs are a natural programming language.

Outline

1. The theory of generable functions
2. A universal differential equation

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \qquad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

Types

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$

Note : existence and unicity of y by Cauchy-Lipschitz theorem.

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Example : $f(x) = x$ ▶ identity

$$y(0) = 0, \qquad y' = 1 \quad \rightsquigarrow \quad y(x) = x$$

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Example : $f(x)=x^2$ ▶ squaring

$$\begin{array}{lll} y_1(0)=0, & y_1'=2y_2 & \rightsquigarrow\ y_1(x)=x^2 \\ y_2(0)=0, & y_2'=1 & \rightsquigarrow\ y_2(x)=x \end{array}$$

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Example : $f(x)=x^n$ ▶ n^{th} power

$$\begin{array}{llll} y_1(0)=0, & y_1'=ny_2 & \rightsquigarrow & y_1(x)=x^n \\ y_2(0)=0, & y_2'=(n-1)y_3 & \rightsquigarrow & y_2(x)=x^{n-1} \\ \dots & \dots & & \dots \\ y_n(0)=0, & y_n=1 & \rightsquigarrow & y_n(x)=x \end{array}$$

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Example : $f(x) = \exp(x)$ ▶ exponential

$$y(0) = 1, \qquad y' = y \quad \rightsquigarrow \quad y(x) = \exp(x)$$

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Example : $f(x) = \sin(x)$ or $f(x) = \cos(x)$ ▸ sine/cosine

$$\begin{aligned} y_1(0) &= 0, & y_1' &= y_2 & \rightsquigarrow \quad y_1(x) &= \sin(x) \\ y_2(0) &= 1, & y_2' &= -y_1 & \rightsquigarrow \quad y_2(x) &= \cos(x) \end{aligned}$$

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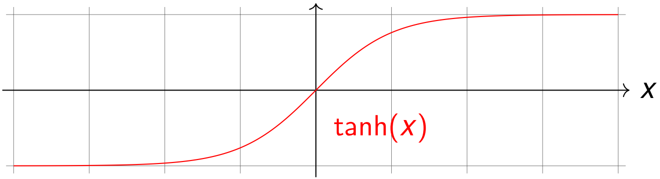
Example : $f(x)=\tanh(x)$ ► hyperbolic tangent

$$y(0)=0, \qquad y'=1-y^2 \quad\rightsquigarrow\quad y(x)=\tanh(x)$$



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Example : $f(x) = \frac{1}{1+x^2}$ ▶ rational function

$$f'(x) = \frac{-2x}{(1+x^2)^2} = -2xf(x)^2$$

$$\begin{aligned} y_1(0) &= 1, & y_1' &= -2y_2y_1^2 & \rightsquigarrow \quad y_1(x) &= \frac{1}{1+x^2} \\ y_2(0) &= 0, & y_2' &= 1 & \rightsquigarrow \quad y_2(x) &= x \end{aligned}$$

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Example : $f = g \pm h$ ▶ sum/difference

$$(f \pm g)' = f' \pm g'$$

assume :

$$\begin{aligned} z(0) &= z_0, & z' &= p(z) & \rightsquigarrow \quad z_1 &= g \\ w(0) &= w_0, & w' &= q(w) & \rightsquigarrow \quad w_1 &= h \end{aligned}$$

then :

$$y(0) = z_{0,1} + w_{0,1}, \qquad y' = p_1(z) \pm q_1(w) \quad \rightsquigarrow \quad y = z_1 \pm w_1$$

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Example : $f = gh$ ▶ product

$$(gh)' = g'h + gh'$$

assume :

$$\begin{array}{lll} z(0) = z_0, & z' = p(z) & \rightsquigarrow \quad z_1 = g \\ w(0) = w_0, & w' = q(w) & \rightsquigarrow \quad w_1 = h \end{array}$$

then :

$$y(0) = z_{0,1} w_{0,1}, \qquad y' = p_1(z) w_1 + z_1 q_1(w) \quad \rightsquigarrow \quad y = z_1 w_1$$

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Example : $f = \frac{1}{g}$ ▶ inverse

$$f' = \frac{-g'}{g^2} = -g' f^2$$

assume :

$$z(0) = z_0, \qquad z' = p(z) \qquad \rightsquigarrow \qquad z_1 = g$$

then :

$$y(0) = \frac{1}{z_{0,1}}, \qquad y' = -p_1(z) y^2 \quad \rightsquigarrow \quad y = \frac{1}{z_1}$$

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Example : $f=\int g$ ▶ integral

assume :

$$z(0)=z_0,\qquad z'=p(z)\quad\rightsquigarrow\quad z_1=g$$

then :

$$y(0)=0,\qquad y'=z_1\quad\rightsquigarrow\quad y=\int z_1$$

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Example : $f=g'$ ▶ derivative

$$f'=g''=(p_1(z))'=\nabla p_1(z)\cdot z'$$

assume :

$$z(0)=z_0,\qquad z'=p(z)\qquad\rightsquigarrow\qquad z_1=g$$

then :

$$y(0)=p_1(z_0),\qquad y'=\nabla p_1(z)\cdot p(z)\qquad\rightsquigarrow\qquad y=z_1''$$

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Example : $f = g \circ h$ ▶ composition

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

assume :

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then :

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Is this coefficient in $\mathbb{K}$?

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Is this coefficient in $\mathbb{K}$? Fields with this property are called generable.

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Example : $f'=\tanh\circ f$ ▶ Non-polynomial differential equation

$$f''=(\tanh'\circ f)f'=(1-(\tanh\circ f)^2)f'$$

$$\begin{array}{lll} y_1(0)=f(0), & y_1'=y_2 & \rightsquigarrow\ y_1(x)=f(x) \\ y_2(0)=\tanh(f(0)), & y_2'=(1-y_2^2)y_2 & \rightsquigarrow\ y_2(x)=\tanh(f(x)) \end{array}$$

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Example : $f(0)=f_0, f'=g\circ f$ ▶ Initial Value Problem (IVP)

$$f'=g''=(p_1(z))'=\nabla p_1(z)\cdot z'$$

assume :

$$z(0)=z_0, \qquad z'=p(z) \qquad \rightsquigarrow \qquad z_1=g$$

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Generable functions : a first summary

Nice theory for the class of total and univariate generable functions :

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$

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Limitations :

- total functions
- univariate

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \qquad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

$J_y(x)$ = Jacobian matrix of y at x

Types

- $n \in \mathbb{N}$: input dimension
- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{K}^{d \times d}[\mathbb{R}^d]$: polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d, y : X \to \mathbb{R}^d$

Notes :

- Partial differential equation !
- Unicity of solution y...
- ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3)$ ▶ monomial

$$y(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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Example : $f(x_1, x_2) = x_1 x_2^2$ ► monomial

$$\begin{array}{llll} y_1(0,0)= 0, & \partial_{x_1} y_1 = y_3^2, & \partial_{x_2} y_1 = 3y_2y_3 & \rightsquigarrow \quad y_1(x) = x_1x_2^2 \\ y_2(0,0)= 0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \rightsquigarrow \quad y_2(x) = x_1 \\ y_3(0,0)= 0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \rightsquigarrow \quad y_3(x) = x_2 \end{array}$$

This is tedious !

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \qquad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

Types

- $n \in \mathbb{N}$: input dimension
- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{K}^{d\times d}[\mathbb{R}^d]$: polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d, y : X \to \mathbb{R}^d$

$J_y(x) =$ Jacobian matrix of y at x

Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$ ▶ inverse function

$$y(1) = 1, \qquad \partial_x y = -y^2 \quad \rightsquigarrow \quad y(x) = \frac{1}{x}$$

Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$

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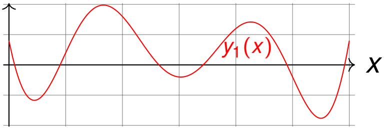
Riemann Γ and ζ are not generable.

Computing with generable functions

Generable functions

$$y(0) = y_0 \qquad y' = p(y)$$

$$f(x) = y_1(x) \qquad x \in \mathbb{R}$$



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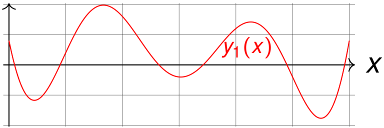

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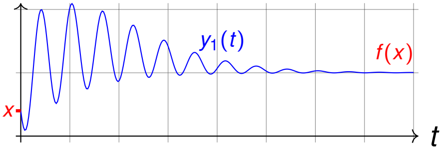
Analog computable function

$$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$$

$$|f(x) - y_1(t)| \leqslant 2^{-t}$$



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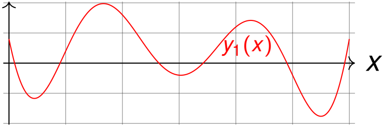

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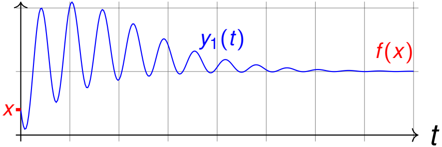

Turing powerful

Theorem (Bournez et al., 2007)

$f : [a, b] \to \mathbb{R}$ is computable[a] iff f is analog computable.

a. In the sense of Computable Analysis.

Computing with generable functions

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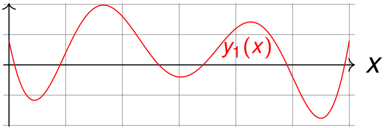

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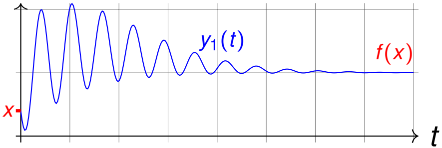

Turing powerful

Question : reformulate analog computability with generable functions ?

Computing with generable functions

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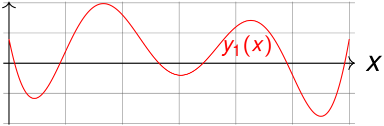

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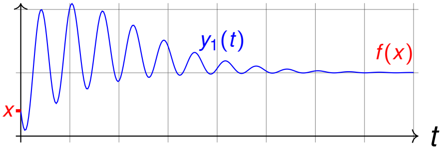

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$f : [a, b] \to \mathbb{R}$ is computable [a] iff $\exists$ a generable function g such that

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- original proof : unclear, something like $\mathbb{Q}(\pi, e, \texttt{others?})$

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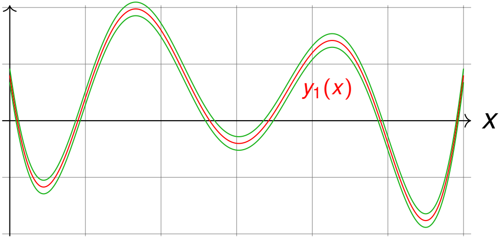
What happens if we take $\mathbb{K} = \mathbb{R}$?

Outline

Universal differential algebraic equation (Rubel)



Theorem (Rubel, 1981)

For any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$\begin{aligned} 3y'^4 y'' y''''^2 \quad & -4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24 y'^2 y''^4 y'''' \\ & -12 y'^3 y'' y'''^3 - 29 y'^2 y''^3 y'''^2 + 12 y''^7 \qquad = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

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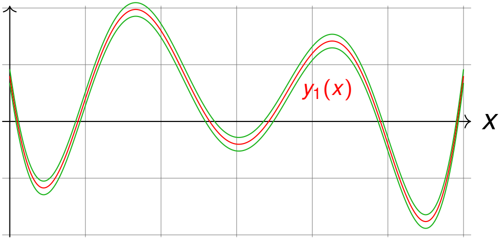
Universal differential algebraic equation (Rubel)



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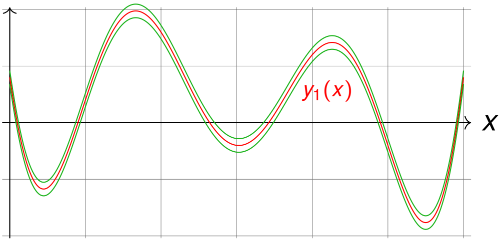

Open Problem (Rubel)

Can we have unicity of the solution with initial conditions ?

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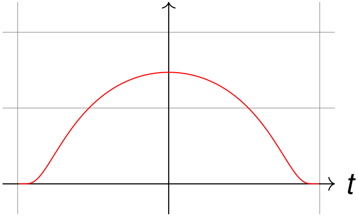
Rubel's ("disappointing") proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

  It satisfies $\quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$



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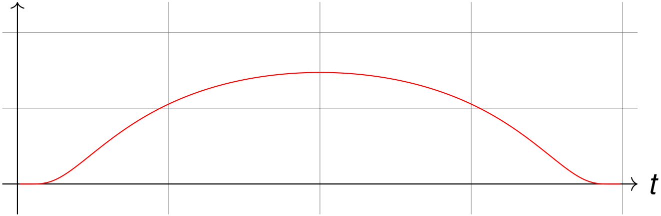
Translation and rescaling :



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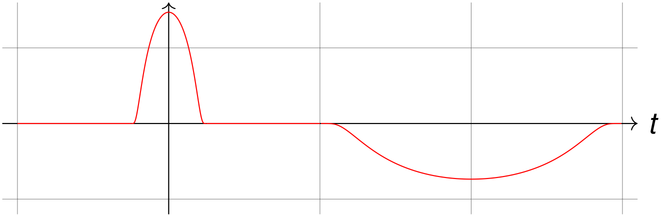
- Can glue together arbitrary many such pieces



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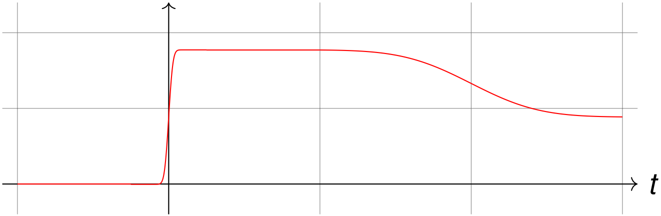
- Can glue together arbitrary many such pieces
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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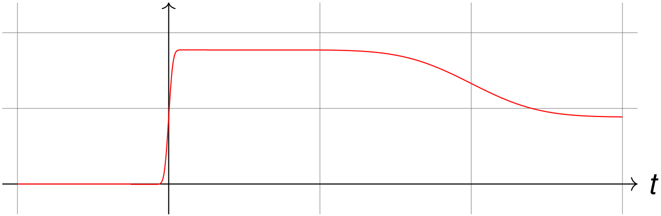

Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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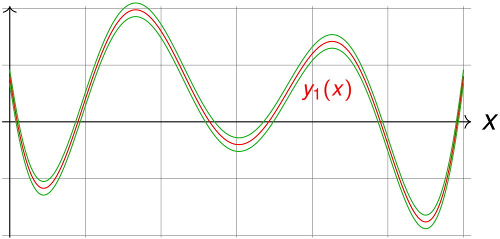
- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Universal explicit ordinary differential equation



Theorem (universal pIVP)

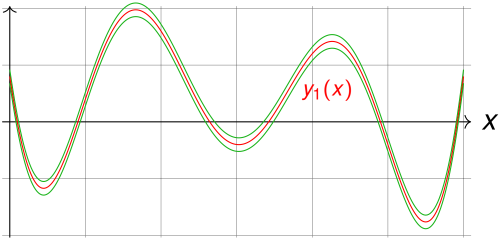
There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Universal explicit ordinary differential equation



Notes :

- **system** of ODEs,
- y must be analytic,
- we need $d \approx 300$.

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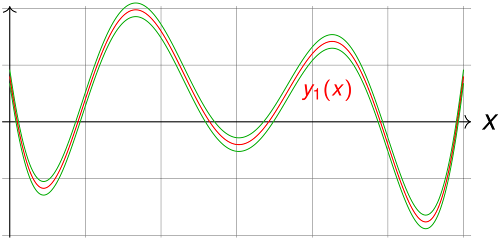


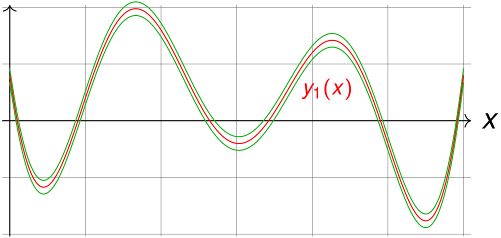
Theorem (universal generable function)

There exists a **fixed** generable function $g :\subseteq \mathbb{R}^{d+1} \to \mathbb{R}$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$|f(t) - g(\alpha, t)| \leqslant \varepsilon(t) \qquad \forall t \in \mathbb{R}.$$

Note : α is usually transcendental, and typically not in $\mathbb{R}_G$...

Universal DAE, again but better



Corollary of main result

There exists a **fixed** k and nontrivial polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha_0, \dots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** $y : \mathbb{R} \to \mathbb{R}$ and $\forall t \in \mathbb{R}$,

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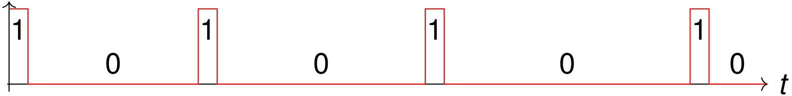
Proof by picture (simplified)

binary stream generator : digits of $\alpha \in \mathbb{R}$



$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2}\tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4,\lambda)} + 4\pi/3))$$

It's horribly generable

round is a mysterious rounding function...

Proof by picture (simplified)

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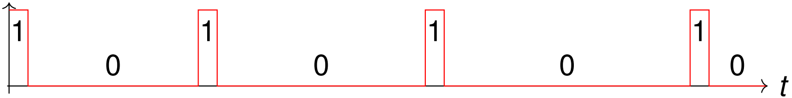

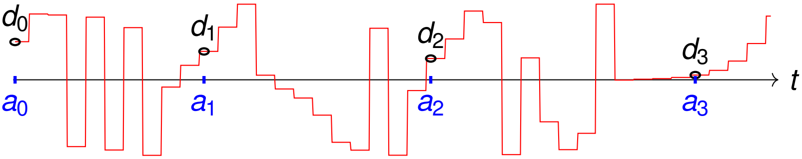

dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j<i} d_j$

$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4,\gamma)}))$$

round is a mysterious rounding function...

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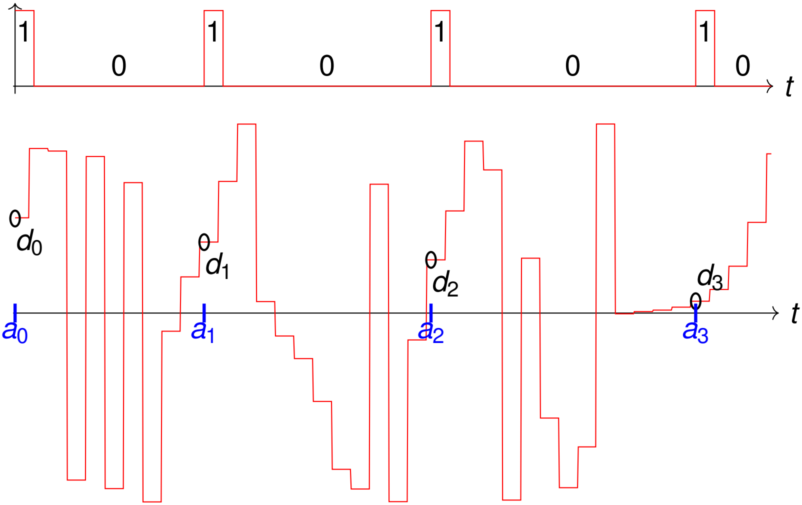

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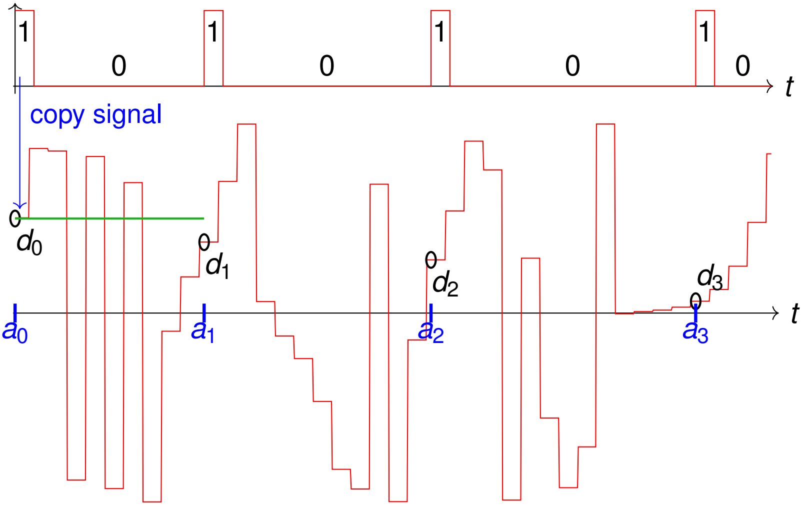

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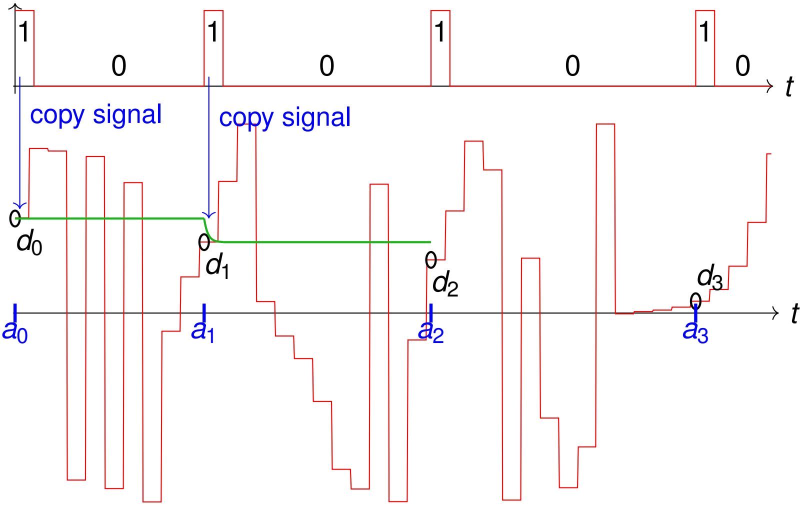
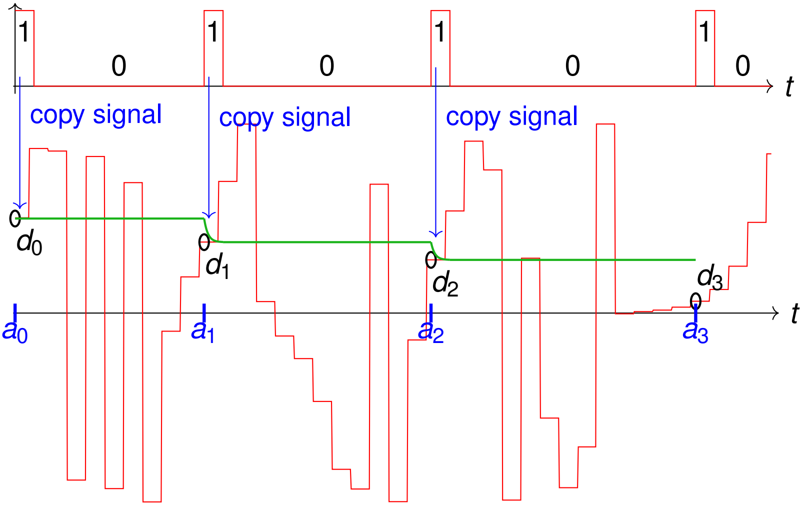


Proof by picture (simplified)



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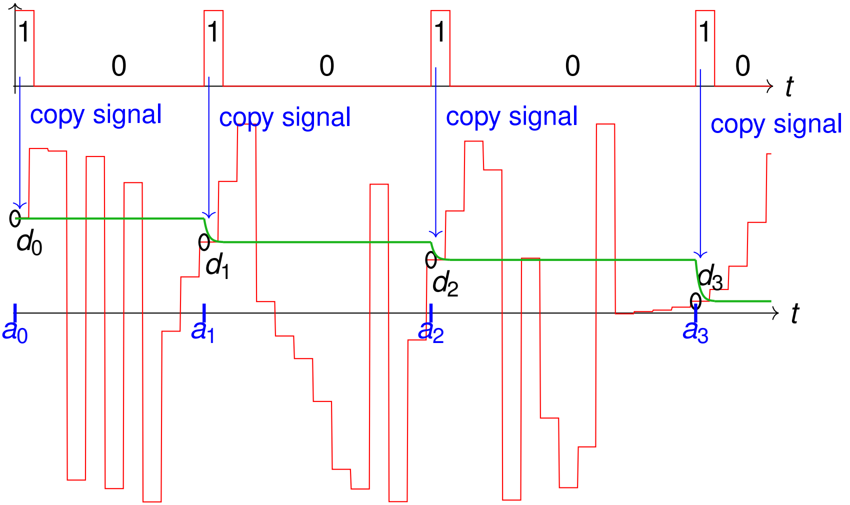

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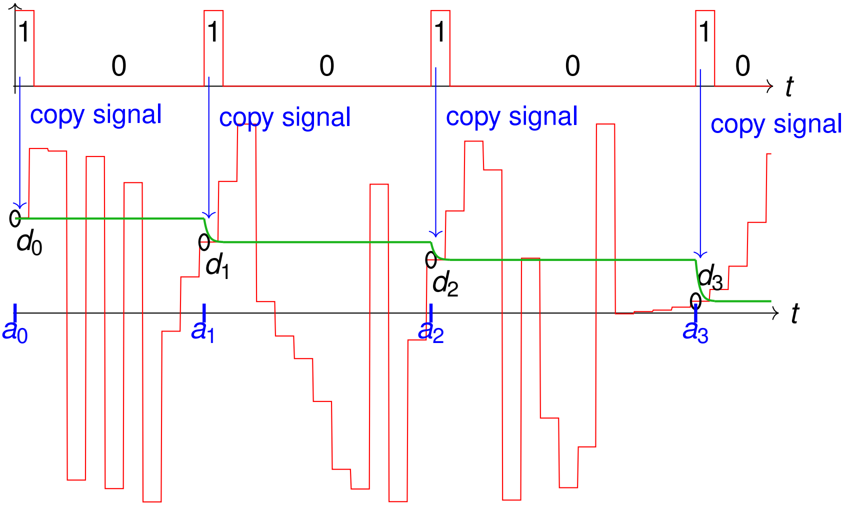
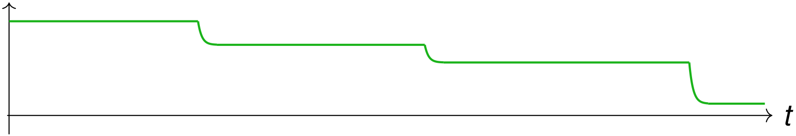


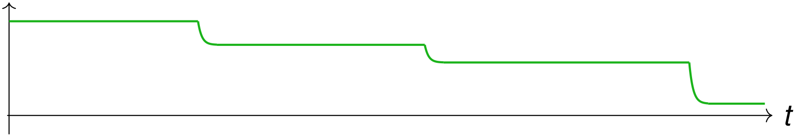
This copy operation is the “non-trivial” part.

Proof by picture (simplified)



We can do **almost piecewise constant functions...**

Proof by picture (simplified)



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

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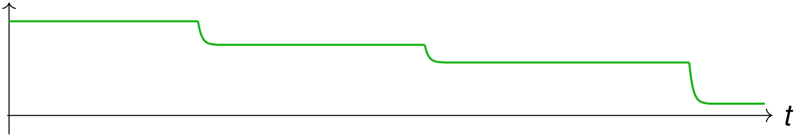

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How do we go to arbitrarily large and growing functions ? Can a polynomial ODE even have arbitrary growth ?

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \qquad \rightsquigarrow \qquad y_1(t) = \exp(t)$$

An old question on growth

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$$\begin{aligned} y_1' &= y_1 & \quad\leadsto\quad & y_1(t) = \exp(t) \\ y_2' &= y_1 y_2 & \quad\leadsto\quad & y_2(t) = \exp(\exp(t)) \end{aligned}$$

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Conjecture (Emile Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

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$e_n(t) = \exp(\cdots \exp(t) \cdots)$ (n compositions)

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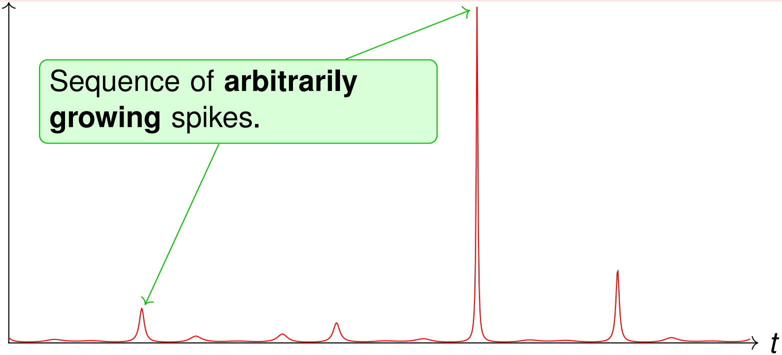
Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



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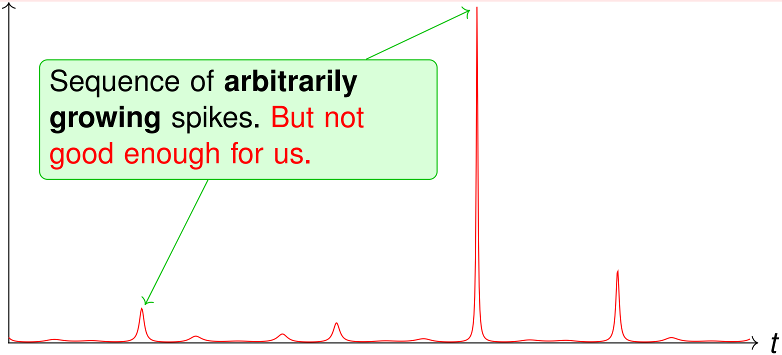

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Theorem (In the paper)

There exists a polynomial $p:\mathbb{R}^d\to\mathbb{R}^d$ such that for any continuous function $f:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$, we can find $\alpha\in\mathbb{R}^d$ such that

$$y(0)=\alpha, \qquad y'(t)=p(y(t))$$

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$$y_1(t)\geqslant f(t), \qquad \forall t\geqslant 0.$$

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Note : both results require α to be **transcendental.** Conjecture still open for **rational** coefficients.

Proof gem : iteration with differential equations

Goal

Iterate f with a GPAC : $y(n) \approx f^{[n]}([x])$

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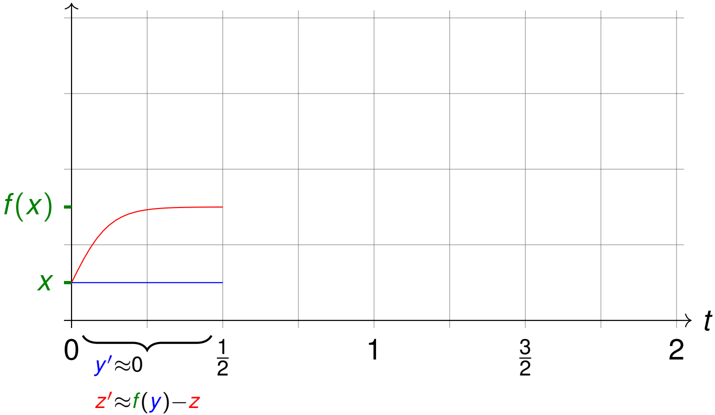

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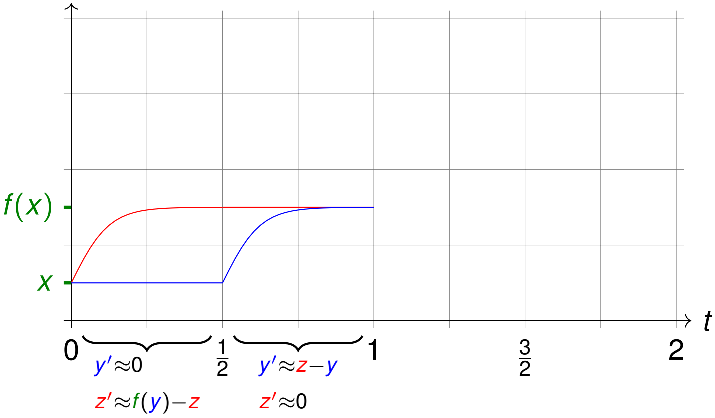

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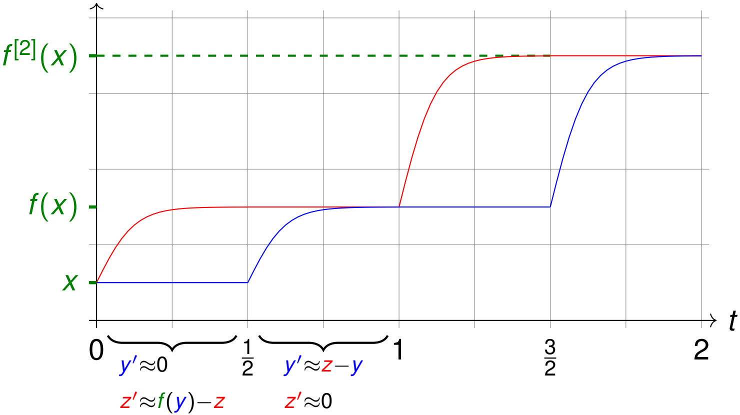

A computability question

Theorem (universal pIVP)

There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

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There exists a **fixed** generable function $g :\subseteq \mathbb{R}^{d+1} \to \mathbb{R}$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$|f(t) - g(\alpha, t)| \leqslant \varepsilon(t) \qquad \forall t \in \mathbb{R}.$$

Question : is α computable from f and ε?

A computability question

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Claim (WIP) : $(f, \varepsilon) \mapsto \alpha$ is computable (for some reasonable representation)

Conclusion

This talk

theory of generable functions

positive answer to Rubel's open problem

Take home

programming with polynomial ODE is nice and fun

Possible development

Each universal ODE defines a map :

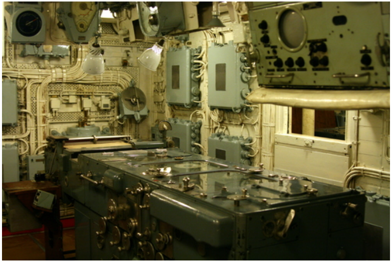
$$(f, \varepsilon) \in C^0 \times C^0 \mapsto \alpha \in \mathbb{R}$$

Kolmogorov-like complexity for continuous functions ?

Digital vs analog computers

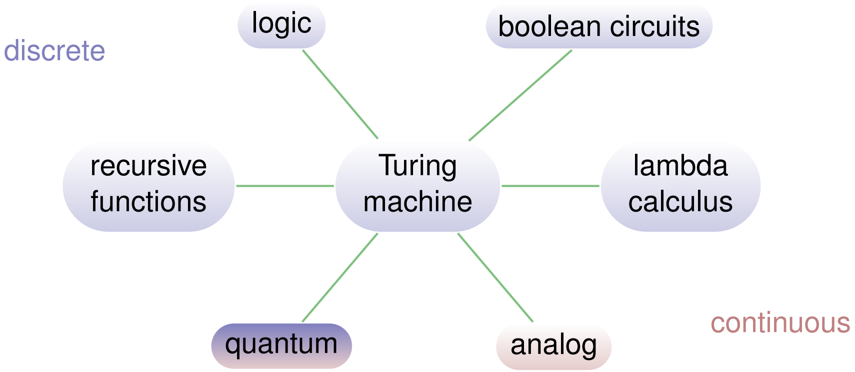
Digital vs analog computers

VS



Church Thesis

Computability

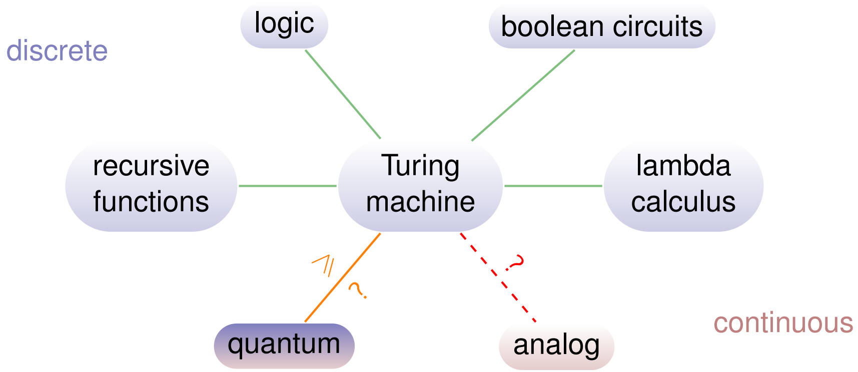


Church Thesis

All reasonable models of computation are equivalent.

Church Thesis

Complexity

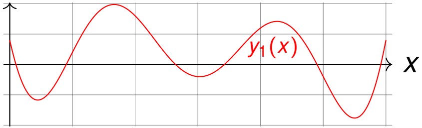


Effective Church Thesis

All reasonable models of computation are equivalent for complexity.

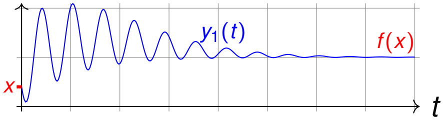
Universal differential equations

Generable functions



subclass of analytic functions

Computable functions



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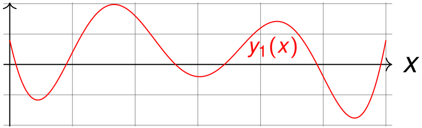

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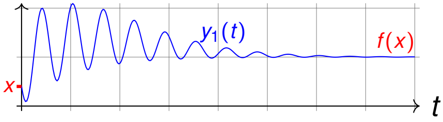

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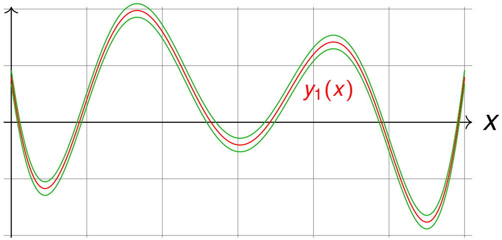

A new notion of computability

Almost-Theorem

$f : [0,1] \to \mathbb{R}$ is **computable** if and only if there exists $\tau > 1$, $y_0 \in \mathbb{R}^d$ and p polynomial such that

$$y'(0) = y_0, \qquad y'(t) = p(y(t))$$

satisfies

$$|f(x) - y(x + n\tau)| \leqslant 2^{-n}, \qquad \forall x \in [0,1], \forall n \in \mathbb{N}$$