Continuous models of computation: computability, complexity, universality

Amaury Pouly

22 august 2017

Teaser

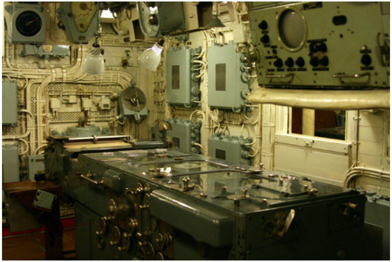
Characterization of P using differential equations

Universal differential equation

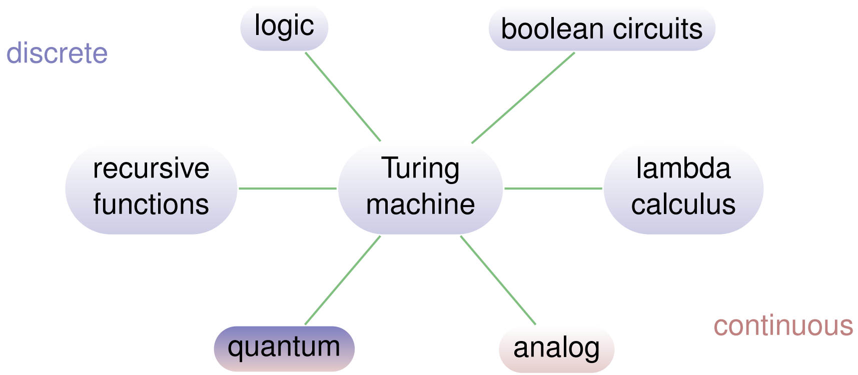
Digital vs analog computers

Digital vs analog computers

VS



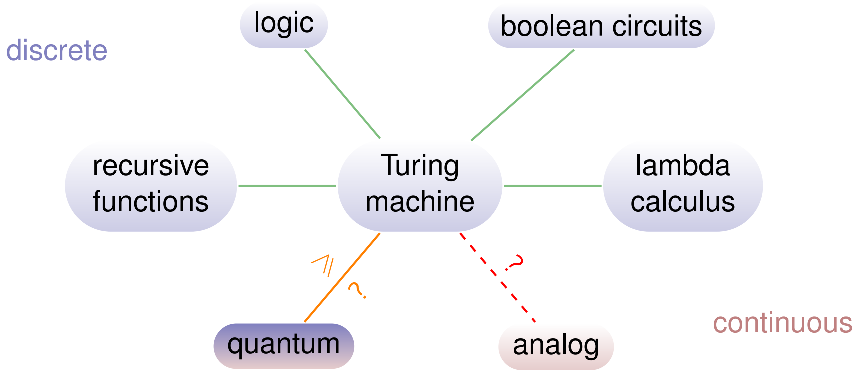
Computability



Church Thesis

All reasonable models of computation are equivalent.

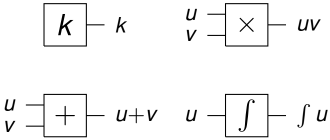
Complexity



Effective Church Thesis

All reasonable models of computation are equivalent for complexity.

Polynomial Differential Equations



General Purpose
Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks :

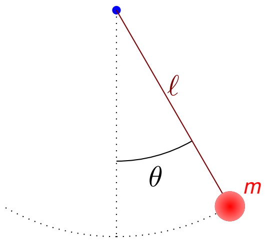
- chemical
- enzymatic

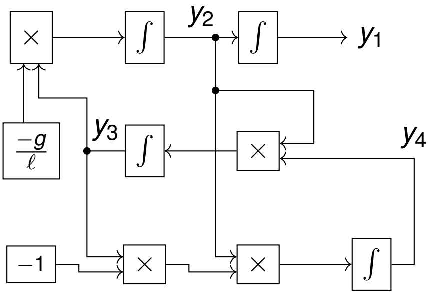
polynomial differential equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable $(+, \times, \circ, /, \text{ED})$
- No closed-form solution

Example of dynamical system





$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

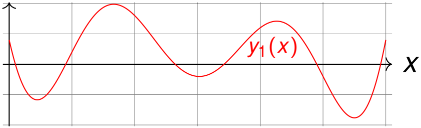
$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{l} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



Shannon's notion

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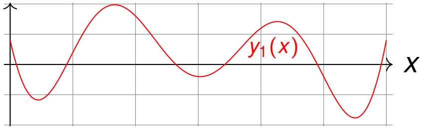

Shannon's notion

sin, cos, exp, log, ...

Strictly weaker than Turing machines [Shannon, 1941]

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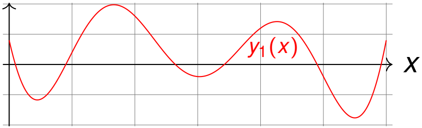

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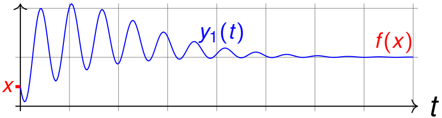
Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$



Modern notion

Computing with differential equations

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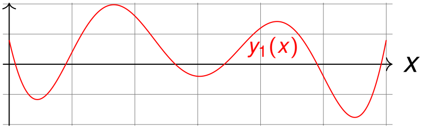

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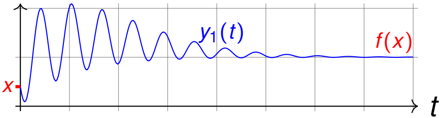


Modern notion

sin, cos, exp, log, Γ, ζ, ...

Turing powerful [Bournez et al., 2007]

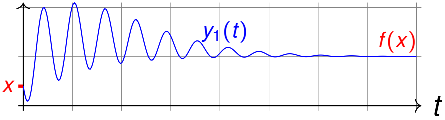
Equivalence with computable analysis

Definition (Bournez et al, 2007)

f computable by GPAC if $\exists p$ polynomial such that $\forall x$

$$y(0) = (x, 0, \ldots, 0) \qquad y'(t) = p(y(t))$$

satisfies $|f(x) - y_1(t)| \leqslant y_2(t)$ et $y_2(t) \xrightarrow[t\to\infty]{} 0$.



$y_1(t) \xrightarrow[t\to\infty]{} f(x)$

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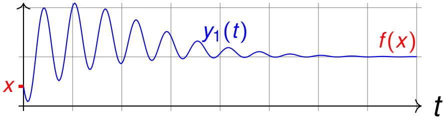

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Theorem (Bournez et al, 2007)

$f : [a, b] \to \mathbb{R}$ computable $\Leftrightarrow$ f computable by GPAC

Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x

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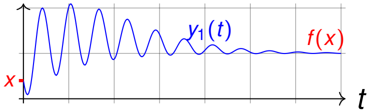
- GPAC : time contraction problem

Tentative definition

$T(x, \mu) =$ first time t so that $|y_1(t) - f(x)| \leqslant e^{-\mu}$

$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$



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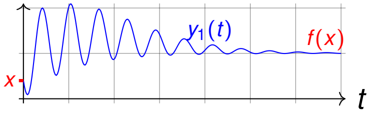
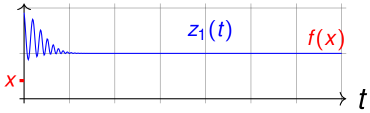
$y(0) = (x, 0, \ldots, 0)$ $\quad y' = p(y)$ $\qquad\qquad z(t) = y(e^t)$



$\rightsquigarrow$



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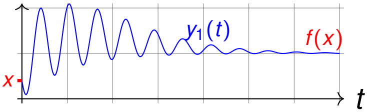

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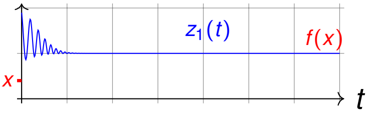
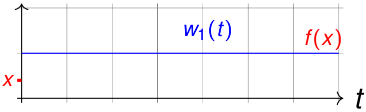


$w(t) = y(e^{e^t})$



Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x
- GPAC : time contraction problem $\rightarrow$ **open problem**

Tentative definition

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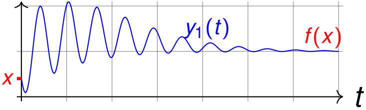

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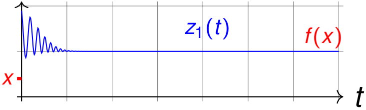

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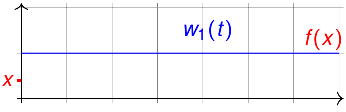
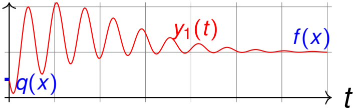


Problem

All functions have constant time complexity.

Time-space correlation of the GPAC

$y(0) = q(x) \qquad y' = p(y)$



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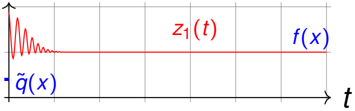

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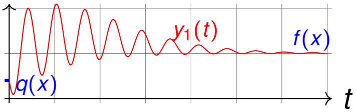

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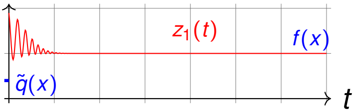
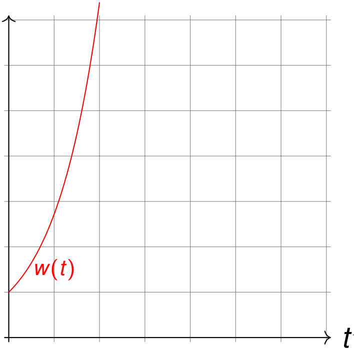
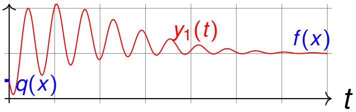
$z(t) = y(e^t)$



extra component : $w(t) = e^t$



Time-space correlation of the GPAC



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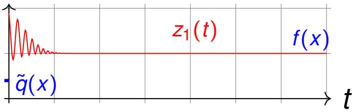

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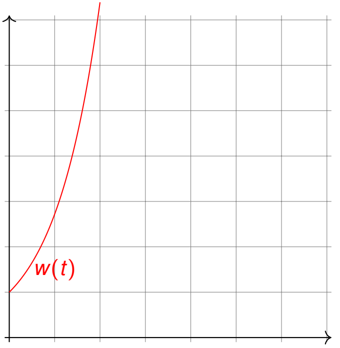


Observation

Time scaling costs "space".

$\rightsquigarrow$

Time complexity for the GPAC must involve time and space !

Complexity of solving polynomial ODEs

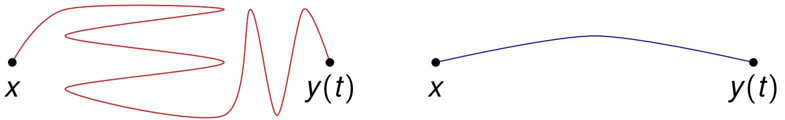
$$y(0) = x \qquad y'(t) = p(y(t))$$

Theorem (Graça, Pouly) [TCS 2016]

If $y(t)$ exists, one can compute p, q such that $\left|\frac{p}{q} - y(t)\right| \leqslant 2^{-n}$ in time

$$\text{poly}(\texttt{size of } x \texttt{ and } p, n, \ell(t))$$

$$\text{where } \ell(t) = \int_0^t \max(1, \|y(u)\|)^{\deg(p)} du \approx \text{length of the curve}$$

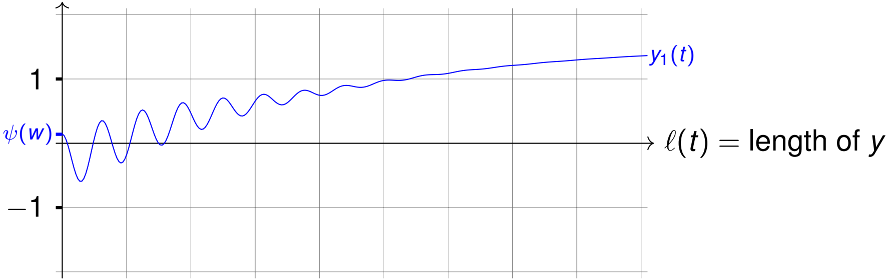


length of the curve = complexity = ressource

Characterization of polynomial time

Definition : $\mathcal{L} \in$ ANALOG-PTIME $\Leftrightarrow \exists p$ polynomial, $\forall$ word w

$$y(0) = (\psi(w), |w|, 0, \ldots, 0) \qquad y' = p(y) \qquad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}$$



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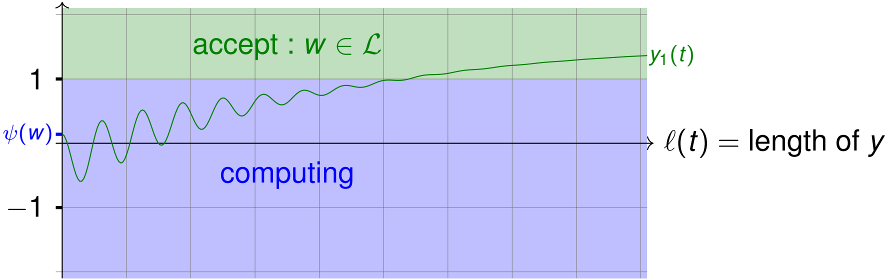

satisfies

1. if $y_1(t) \geqslant 1$ then $w \in \mathcal{L}$

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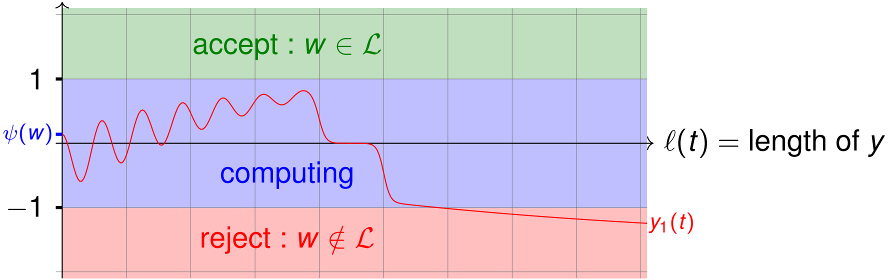

satisfies

2. if $y_1(t) \leqslant -1$ then $w \notin \mathcal{L}$

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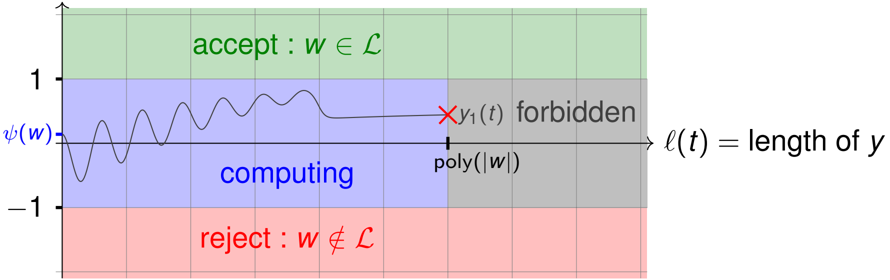

satisfies

3. if $\ell(t) \geqslant \text{poly}(|w|)$ then $|y_1(t)| \geqslant 1$

Characterization of polynomial time

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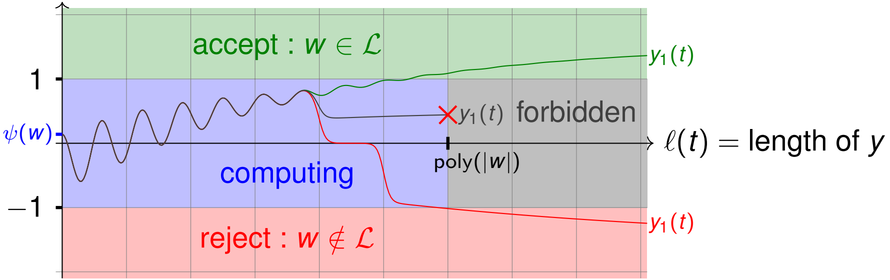


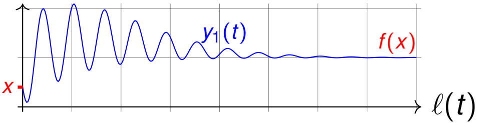
Theorem (JoC 2016 ; ICALP 2016)

PTIME = ANALOG-PTIME

Characterization of real polynomial time

Definition : $f : [a,b] \to \mathbb{R}$ in ANALOG-P$_\mathbb{R}$ $\Leftrightarrow$ $\exists p$ polynomial, $\forall x \in [a,b]$

$$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$$



Characterization of real polynomial time

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$$y(0)=(x,0,\ldots,0)\qquad y'=p(y)$$

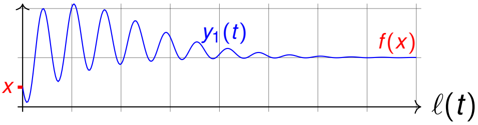
satisfies :

1. $|y_1(t)-f(x)|\leqslant 2^{-\ell(t)}$

   «greater length $\Rightarrow$ greater precision»

2. $\ell(t)\geqslant t$

   «length increases with time»



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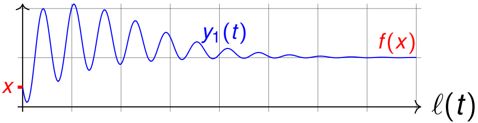
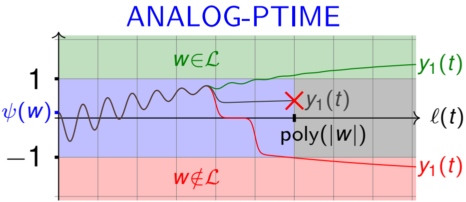
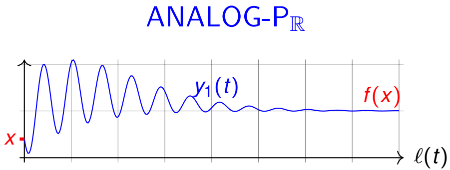


Theorem (JoC 2016 ; ICALP 2016)

$f : [a,b] \to \mathbb{R}$ computable in polynomial time $\Leftrightarrow f \in$ ANALOG-P$_{\mathbb{R}}$.

Summary



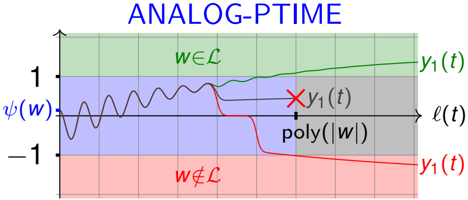


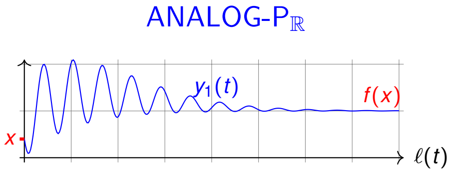
Theorem [JoC 2016 ; ICALP 2016]

- $\mathcal{L} \in$ PTIME of and only if $\mathcal{L} \in$ ANALOG-PTIME
- $f : [a, b] \to \mathbb{R}$ computable in polynomial time $\Leftrightarrow f \in$ ANALOG-$P_{\mathbb{R}}$

- Analog complexity theory based on length
- Time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME

Summary





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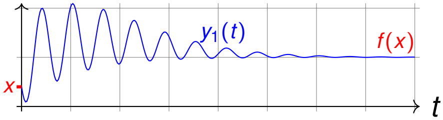
- Analog complexity theory based on length
- Time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME
- Only rational coefficients needed (JACM 2017)

Universal differential equations

Generable functions

subclass of analytic functions

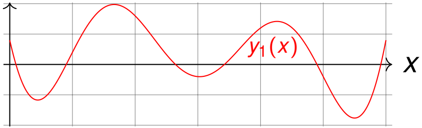
Computable functions



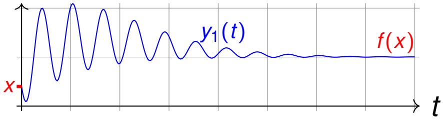
any computable function

Universal differential equations

Generable functions

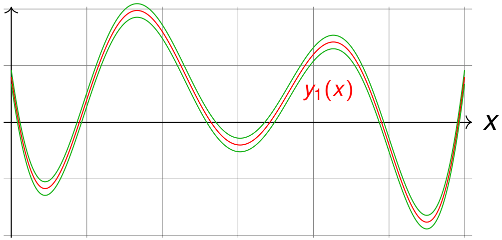


Computable functions

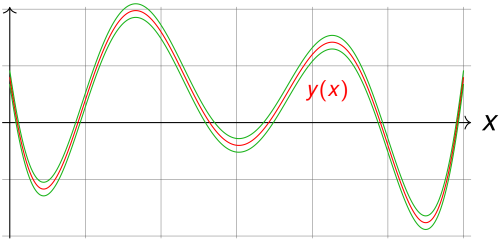


subclass of analytic functions

any computable function



Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

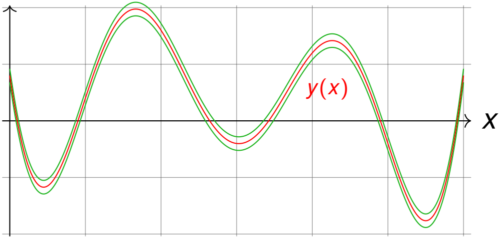
For any continuous functions f and ε, there exists $y : \mathbb{R} \to \mathbb{R}$ solution to

$$\begin{aligned} 3y'^4 y'' y''''^2 \quad & -4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ & -12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 \qquad = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$p(y, y', \ldots, y^{(k)}) = 0$$

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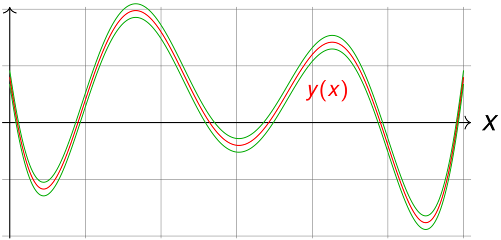

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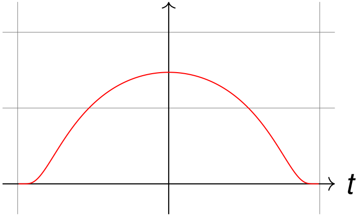
Problem : this is «weak» result.

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

$$\text{It satisfies} \quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$$



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- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at+b)$ satisfies

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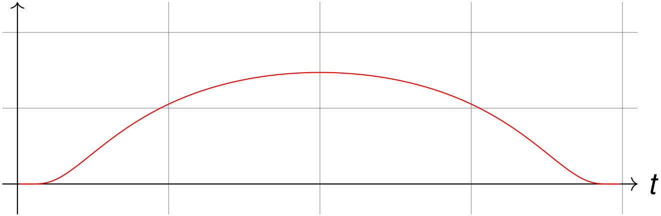
Translation and rescaling :



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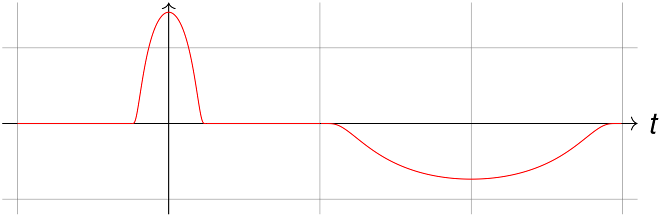
- Can glue together arbitrary many such pieces



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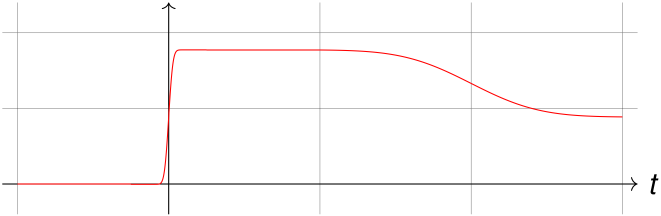
- Can glue together arbitrary many such pieces
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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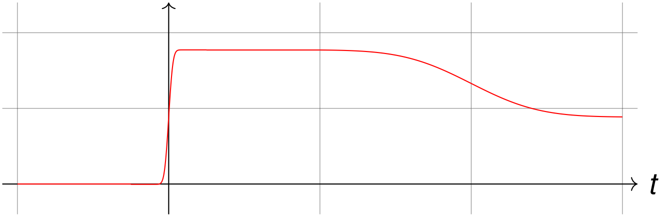

Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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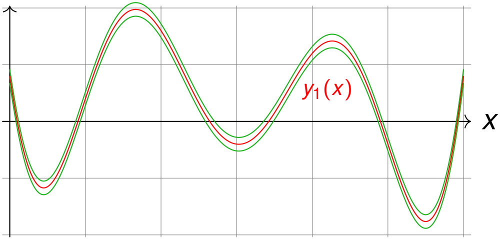
In fact, this is fundamental for Rubel's proof to work !

- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Universal initial value problem (IVP)



Notes :

- **system** of ODEs,
- y is analytic,
- we need $d \approx 300$.

Theorem (ICALP 2017)

There exists a **fixed** (vector of) polynomial p such that for any continuous functions f and ε, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Note : α is usually transcendental, but computable from f and ε

Future work

[CMSB17] ⟷ Reaction networks :

- chemical
- enzymatic

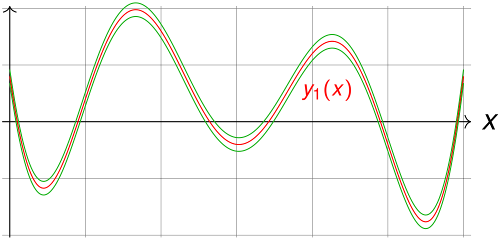
$$y' = p(y)$$

?

$$y' = p(y) + e(t)$$

- Finer time complexity (linear)
- Nondeterminism
- Robustness
- « space» complexity
- Other models
- Stochastic

Universal DAE revisited



Theorem

There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists $\alpha_0, \dots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** and this solution satisfies such that

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$