A (truly) universal differential equation

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Joint work with Olivier Bournez and Daniel Graça

Travel supported by NSF DMS-1952694

14 February 2020
What is a computer?
What is a computer?
What is a computer?
Church Thesis

All **reasonable** models of computation are equivalent.
Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.
Polynomial Differential Equations

Differential Analyzer

Reaction networks:
- chemical
- enzymatic

Newton mechanics
Polynomial Differential Equations

\[ k \quad \quad u \quad \quad \times \quad \quad uv \]

\[ u \quad \quad + \quad \quad u + v \quad \quad u \quad \quad \int \quad \quad \int u \]

General Purpose Analog Computer, Shannon 1936
Polynomial Differential Equations

\[ k \quad u \quad \times \quad uv \]

\[ u \quad v \quad + \quad u+v \quad u \quad \int \quad \int u \]

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Differential Analyzer

Polynomial differential equations:
\[
\begin{align*}
y(0) &= y_0 \\
y'(t) &= p(y(t))
\end{align*}
\]

- Rich class
- Stable (+,×,⊙,/,ED)
- No closed-form solution
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Example of dynamical system

\[ \ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \]
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\[
\begin{align*}
  y_1' &= y_2 \\
  y_2' &= -\frac{g}{\ell} y_3 \\
  y_3' &= y_2 y_4 \\
  y_4' &= -y_2 y_3
\end{align*}
\]

\[
\begin{align*}
  y_1 &= \theta \\
  y_2 &= \dot{\theta} \\
  y_3 &= \sin(\theta) \\
  y_4 &= \cos(\theta)
\end{align*}
\]

Historical remark: the word “analog.”

The pendulum and the circuit have the same equation. One can study one using the other by analogy.
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The pendulum and the circuit have the same equation. One can study one using the other by analogy.
Generable functions

\[
\begin{cases}
  y(0) = y_0 \\
  y'(x) = p(y(x))
\end{cases} \quad x \in \mathbb{R}
\]

\[f(x) = y_1(x)\]

Shannon’s notion
Generable functions

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Shannon’s notion

\[\sin, \cos, \exp, \log, \ldots\]

Considered "weak" : not \(\Gamma\) and \(\zeta\)

Only analytic functions
Does a balance scale compute a function?

Inputs: $x, y \in [0, +\infty)$

Output: $\text{sign}(x - y)$
Does a balance scale compute a function?

**Inputs:** $x, y \in [0, +\infty)$

\[ x = y \]
Does a balance scale compute a function?

**Inputs:** $x, y \in [0, +\infty)$

\[ x > y \]

\[ x = y \]
Does a balance scale compute a function?

**Inputs:** \( x, y \in [0, +\infty) \)

- **Output:** \( \text{sign}(x - y) \)
Does a balance scale compute a function?

Inputs: \( x, y \in [0, +\infty) \)

Output: \( \text{sign}(x - y) \)?
More formally

Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.
More formally

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More formally

Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

- analog computability theory
- purely continuous characterization of classical computability
Generable functions

\[
\begin{cases}
  y(0) = y_0 \\
y'(x) = p(y(x))
\end{cases} \quad x \in \mathbb{R}
\]

\[f(x) = y_1(x)\]

Computable

\[
\begin{cases}
  y(0) = q(x) \\
y'(t) = p(y(t))
\end{cases} \quad x \in \mathbb{R}
\]

\[f(x) = \lim_{t \to \infty} y_1(t)\]

Shannon’s notion

\[\sin, \cos, \exp, \log, \ldots\]

Considered "weak": not \(\Gamma\) and \(\zeta\)

Only analytic functions

Modern notion

\[\sin, \cos, \exp, \log, \Gamma, \zeta, \ldots\]

Turing powerful

[Bournez et al., 2007]
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Theorem (Rubel, 1981)

For any continuous functions \( f \) and \( \varepsilon \), there exists \( y : \mathbb{R} \rightarrow \mathbb{R} \) solution to

\[
3y^4 y'' y''''^2 - 4y'^4 y'''^2 y'''' + 6y'^3 y'' y''' y'''' + 24y'^2 y'''^4 y''''
- 12y'^3 y'' y'''^3 - 29y'^2 y'''^3 y''''^2 + 12y''^7 = 0
\]

such that \( \forall t \in \mathbb{R}, \)

\[
|y(t) - f(t)| \leq \varepsilon(t).
\]
Theorem (Rubel, 1981)

There exists a fixed polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$p(y, y', \ldots, y^{(4)}) = 0$$

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such that \( \forall t \in \mathbb{R}, \)

\[
|y(t) - f(t)| \leq \varepsilon(t).
\]

Problem: this is «weak» result.
The problem with Rubel’s DAE

The solution $y$ is not unique, even with added initial conditions:

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel’s proof to work!
The problem with Rubel’s DAE

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- Rubel’s statement: this DAE is universal
- More realistic interpretation: this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?

Note: explicit polynomial ODE $\Rightarrow$ unique solution
Rubel’s proof in one slide

Take \( f(t) = e^{\frac{-1}{1-t^2}} \) for \(-1 < t < 1\) and \(f(t) = 0\) otherwise.

It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0\).
Rubel’s proof in one slide

- Take \( f(t) = e^{\frac{1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise.

\[
\text{It satisfies } (1 - t^2)^2 f''(t) + 2tf'(t) = 0.
\]

- For any \( a, b, c \in \mathbb{R}, y(t) = cf(at + b) \) satisfies

\[
3y''^4 y''''^2 - 4y''^4 y''''^2 y'''' + 6y'''^2 y'' y'''' + 24y'^2 y'''^4 y'''' + 12y''''^7 = 0
\]

Translation and rescaling:
Rubel’s proof in one slide

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  3y'^4y''^{\prime\prime\prime\prime} - 4y'^4y''^{\prime\prime\prime} + 6y'^3y''^{\prime\prime} + 24y'^2y''^{\prime\prime}y'' - 12y'^3y''^{\prime\prime} - 29y'^2y''^2 + 12y''^7 = 0
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- Can glue together arbitrary many such pieces
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- Can glue together arbitrary many such pieces

- Can arrange so that $\int f$ is solution: \textbf{piecewise pseudo-linear}

Conclusion: Rubel’s equation allows any piecewise pseudo-linear functions, and those are \textbf{dense in $C^0$}
Universal initial value problem (IVP)

Theorem

There exists a fixed (vector of) polynomial $p$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a unique solution $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$
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Notes:

- system of ODEs,
- $y$ is analytic,
- we need $d \approx 300$. 

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**Remark**: $\alpha$ is usually transcendental, but computable from $f$ and $\varepsilon$.
Theorem

There exists a **fixed** polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, \ y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** and this solution satisfies such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by programming with ODEs.
Generable functions: a summary

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if \( \exists \, d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
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satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).
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satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

Nice theory for the class of total and univariate \textbf{generable} functions:

- analytic
- contains polynomials, \( \sin, \cos, \tanh, \exp \)
- stable under \( \pm, \times, /, \circ \) and Initial Value Problems (IVP)

\[
y' = f(y)
\]

- solutions to polynomial ODEs form a \textbf{very large class}
Why is this useful?

Writing polynomial ODEs by hand is hard.
Why is this useful?

Writing polynomial ODEs by hand is hard.

Using generable functions, we can build complicated multivariate partial functions using other operations, and we know they are solutions to polynomial ODEs by construction.

Example: almost rounding function

There exists a generable function \( \text{round} \) such that for any \( n \in \mathbb{Z}, x \in \mathbb{R}, \lambda > 2 \) and \( \mu \geq 0 \):

\[ \text{if } x \in [n - \frac{1}{2}, n + \frac{1}{2}] \text{ then } |\text{round}(x, \mu, \lambda) - n| \leq \frac{1}{2}, \]

\[ \text{if } x \in [n - \frac{1}{2} + 1\lambda, n + \frac{1}{2} - 1\lambda] \text{ then } |\text{round}(x, \mu, \lambda) - n| \leq e^{-\mu}. \]
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Writing polynomial ODEs by hand is hard.

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Example: almost rounding function

There exists a generable function round such that for any \( n \in \mathbb{Z}, x \in \mathbb{R}, \lambda > 2 \) and \( \mu \geq 0 \):

- if \( x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right] \) then \( |\text{round}(x, \mu, \lambda) - n| \leq \frac{1}{2} \),
- if \( x \in \left[n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}\right] \) then \( |\text{round}(x, \mu, \lambda) - n| \leq e^{-\mu} \).
A simplified proof

NOTE

This is the ideal curve, the real one is an approximation of it.
A simplified proof

\[ \alpha \in \mathbb{R} \rightarrow \text{ODE} \rightarrow 0 \; 1 \; 1 \; 0 \; 1 \; 0 \; 1 \; 0 \; 0 \; 1 \; 1 \; 1 \; \ldots \rightarrow t \]

Approximate Lipschitz and bounded functions with fixed precision.

That's the trickiest part.
A simplified proof

We need something more: a **fast-growing** ODE.
We need something more: an **arbitrarily fast-growing** ODE.
A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$

\[
f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2} \tanh(\mu \sin(2\alpha \pi 4^{\text{round}(t-1/4, \lambda)} + 4\pi/3))
\]

It’s horrible, but generable

round is the mysterious rounding function...
A less simplified proof

binary stream generator: digits of $\alpha \in \mathbb{R}$

\[ d_i = m_i 2^{-d_i}, \quad a_i = 9i + \sum_{j<i} d_j \]

\[ f(\alpha, \gamma, t) = \sin(2\alpha \pi 2^{\text{round}(t-1/4, \gamma)}) \]

\text{round is the mysterious rounding function...}
A less simplified proof
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A less simplified proof

This copy operation is the “non-trivial” part.
We can do almost piecewise constant functions...
We can do almost piecewise constant functions...  
► ...that are bounded by 1...  
► ...and have super slow changing frequency.
A less simplified proof

We can do almost piecewise constant functions...

- ...that are bounded by 1...
- ...and have super slow changing frequency.

How do we go to arbitrarily large and growing functions? Can a polynomial ODE even have arbitrary growth?
An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \quad \sim \quad y_1(t) = \exp(t)$$
Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[
\begin{align*}
   y'_1 &= y_1 &\implies & y_1(t) = \exp(t) \\
   y'_2 &= y_1 y_2 &\implies & y_1(t) = \exp(\exp(t))
\end{align*}
\]
An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[
\begin{align*}
y_1' &= y_1 \\
y_2' &= y_1 y_2 \\
\cdots \\
y_n' &= y_1 \cdots y_n
\end{align*}
\]

\[
\begin{align*}
y_1(t) &= \exp(t) \\
y_1(t) &= \exp(\exp(t)) \\
\cdots \\
y_n(t) &= \exp(\cdots \exp(t) \cdots) := e_n(t)
\end{align*}
\]
Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[
\begin{align*}
    y_1' &= y_1 \quad \leadsto \quad y_1(t) = \exp(t) \\
    y_2' &= y_1 y_2 \quad \leadsto \quad y_1(t) = \exp(\exp(t)) \\
    \vdots & \quad \vdots \\
    y_n' &= y_1 \cdots y_n \quad \leadsto \quad y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t)
\end{align*}
\]

Conjecture (Emil Borel, 1899)

With $n$ variables, cannot do better than $O_t(e_n(At^k))$. 
An old question on growth

Counter-example (Vijayaraghavan, 1932)

\[
\frac{1}{2 - \cos(t) - \cos(\alpha t)}
\]

Sequence of arbitrarily growing spikes.
An old question on growth

Counter-example (Vijayaraghavan, 1932)

\[ \frac{1}{2 - \cos(t) - \cos(\alpha t)} \]

Sequence of \textbf{arbitrarily growing} spikes. But not good enough for us.
An old question on growth

**Theorem**

*There exists a polynomial* $p : \mathbb{R}^d \to \mathbb{R}^d$ *such that for any continuous function* $f : \mathbb{R}_+ \to \mathbb{R}$, *we can find* $\alpha \in \mathbb{R}^d$ *such that*

$satisfies$

\[ y(0) = \alpha, \quad y'(t) = p(y(t)) \]

\[ y_1(t) \geq f(t), \quad \forall t \geq 0. \]
An old question on growth

**Theorem**

There exists a polynomial $p : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

\[
y(0) = \alpha, \quad y'(t) = p(y(t))
\]

satisfies

\[
y_1(t) \geq f(t), \quad \forall t \geq 0.
\]

**Note**: both results require $\alpha$ to be **transcendental**. Conjecture still open for **rational** (or algebraic) coefficients.
Assume $f$ is generable, can we iterate $f$ with an ODE?
That is, build a generable $y$ such that $y(x, n) \approx f^n(x)$ for all $n \in \mathbb{N}$.
Assume $f$ is generable, can we iterate $f$ with an ODE? That is, build a generable $y$ such that $y(x, n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$.
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Theorem

There exists a fixed (vector of) polynomial \( p \) such that for any continuous functions \( f \) and \( \varepsilon \), there exists \( \alpha \in \mathbb{R}^d \) such that

\[
y(0) = \alpha, \quad y'(t) = p(y(t))
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has a unique solution \( y : \mathbb{R} \to \mathbb{R}^d \) and \( \forall t \in \mathbb{R} \),

\[
|y_1(t) - f(t)| \leq \varepsilon(t).
\]

Remark: \( \alpha \) is usually transcendental, but computable from \( f \) and \( \varepsilon \)

Notes:
- system of ODEs,
- \( y \) is analytic,
- we need \( d \approx 300 \).
Theorem

There exists a fixed polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

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Backup slides
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is generable if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{R}^d[\mathbb{R}^n]$: polynomial vector
- $y_0 \in \mathbb{R}^d$, $y : \mathbb{R} \rightarrow \mathbb{R}^d$

**Note**: existence and unicity of $y$ by Cauchy-Lipschitz theorem.
### Generable functions (total, univariate)

#### Definition

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#### Types

- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{R}^d[\mathbb{R}^n]$: polynomial vector
- $y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d$

#### Example

$f(x) = x$  ➤  identity

$$y(0) = 0, \quad y' = 1 \quad \leadsto \quad y(x) = x$$
Generable functions (total, univariate)

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\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Example:** \( f(x) = x^2 \)  

\[
\begin{align*}
y_1(0) &= 0, \quad y'_1 = 2y_2 \\
y_2(0) &= 0, \quad y'_2 = 1
\end{align*}
\]

\( \leadsto \)

\[
\begin{align*}
y_1(x) &= x^2 \\
y_2(x) &= x
\end{align*}
\]

**Types**

- \( d \in \mathbb{N} \): dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \): polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)
**Generable functions (total, univariate)**

<table>
<thead>
<tr>
<th>Definition</th>
<th>Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f : \mathbb{R} \to \mathbb{R} ) is generable if there exists ( d, p ) and ( y_0 ) such that the solution ( y ) to ( y(0) = y_0, \ y'(x) = p(y(x)) ) satisfies ( f(x) = y_1(x) ) for all ( x \in \mathbb{R} ).</td>
<td>▶ ( d \in \mathbb{N} ) : dimension&lt;br&gt;▶ ( p \in \mathbb{R}^d[\mathbb{R}^n] ) : polynomial vector&lt;br&gt;▶ ( y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d )</td>
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**Example**: \( f(x) = x^n \)  

\[
\begin{align*}
y_1(0) &= 0, & y_1' &= ny_2 \\
y_2(0) &= 0, & y_2' &= (n - 1)y_3 \\
&\vdots & & \vdots \\
y_n(0) &= 0, & y_n &= 1
\end{align*}
\]

\( \leadsto \) \( y_1(x) = x^n \)

\( \leadsto \) \( y_2(x) = x^{n-1} \)

\( \leadsto \) \( y_n(x) = x \)
Generable functions (total, univariate)

Definition

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is \textbf{generable} if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]
satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

Types

\begin{itemize}
  \item \( d \in \mathbb{N} \): dimension
  \item \( p \in \mathbb{R}^d[\mathbb{R}^n] \): polynomial vector
  \item \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)
\end{itemize}

Example: \( f(x) = \exp(x) \) \hspace{1cm} \textbf{exponential}

\[
y(0) = 1, \quad y' = y \quad \sim \quad y(x) = \exp(x)
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is *generable* if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
  y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \): dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \): polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d \)
- sine/cosine

**Example**

\( f(x) = \sin(x) \) or \( f(x) = \cos(x) \)

\[
  y_1(0) = 0, \quad y_1' = y_2 \quad \sim \quad y_1(x) = \sin(x)
\]
\[
  y_2(0) = 1, \quad y_2' = -y_1 \quad \sim \quad y_2(x) = \cos(x)
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \) : polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example** : \( f(x) = \tanh(x) \)  

\[
y(0) = 0, \quad y' = 1 - y^2 \quad \Rightarrow \quad y(x) = \tanh(x)
\]
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{R}^d[\mathbb{R}^n]$: polynomial vector
- $y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

**Example:** $f(x) = \frac{1}{1+x^2} \quad \Rightarrow \text{rational function}$

$$f'(x) = -\frac{2x}{(1+x^2)^2} = -2xf(x)^2$$

$$y_1(0)= 1, \quad y_1' = -2y_2y_1^2 \quad \Rightarrow \quad y_1(x) = \frac{1}{1+x^2}$$

$$y_2(0)= 0, \quad y_2' = 1 \quad \Rightarrow \quad y_2(x) = x$$
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example**: $f = g \pm h$  

**sum/difference**

$$(f \pm g)' = f' \pm g'$$

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Generable functions (total, univariate)

**Definition**

\[ f : \mathbb{R} \rightarrow \mathbb{R} \text{ is generable if there exists } d, p \text{ and } y_0 \text{ such that the solution } y \text{ to } \]
\[ y(0) = y_0, \quad y'(x) = p(y(x)) \]
\[ \text{satisfies } f(x) = y_1(x) \text{ for all } x \in \mathbb{R}. \]

**Types**

- \( d \in \mathbb{N} : \text{dimension} \)
- \( p \in \mathbb{R}^d[\mathbb{R}^n] : \text{polynomial vector} \)
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example :** \( f = gh \)  

\[ (gh)' = g'h + gh' \]
**Generable functions (total, univariate)**

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
\begin{align*}
    y(0) &= y_0, \\
    y'(x) &= p(y(x))
\end{align*}
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Example:** \( f = \frac{1}{g} \)  

\[
    f' = \frac{-g'}{g^2} = -g'f^2
\]

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Generable functions (total, univariate)

Definition

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\[
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\]
satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

Example: \( f = \int g \) ★ integral

\[
f' = g
\]

Types

- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \) : polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d \)
Generable functions (total, univariate)

**Definition**

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**Types**

- \( d \in \mathbb{N} \): dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \): polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example**: \( f = g' \) — derivative

\[
f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]
satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \) : polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example:** \( f = g \circ h \)  

\[
(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \): dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \): polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d \)

**Example:** \( f' = \tanh \circ f \)  

Non-polynomial differential equation

\[
f'' = (\tanh' \circ f)f' = (1 - (\tanh \circ f)^2)f'
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{R}^d[\mathbb{R}^n] \) : polynomial vector
- \( y_0 \in \mathbb{R}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

**Example**:

\( f(0) = f_0, f' = g \circ f \)

**Initial Value Problem (IVP)**

\[
f' = g'' = (p(z))' = \nabla p(z) \cdot z'
\]
Generable functions: a first summary

Nice theory for the class of total and univariate generable functions:
- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
Generable functions: a first summary

Nice theory for the class of total and univariate generable functions:

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- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$

Limitations:

- total functions
- univariate
Generable functions (generalization)

**Definition**

\( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is **generable** if \( X \) is open **connected** and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) = \) Jacobian matrix of \( y \) at \( x \)

**Notes**:

- Partial differential equation!
- Unicity of solution \( y \)...
- ... **but not existence** (ie you have to show it exists)

**Types**

- \( n \in \mathbb{N} : \) input dimension
- \( d \in \mathbb{N} : \) dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] : \) polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)
Generable functions (generalization)

Definition

\( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is generable if \( X \) is open and connected and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) \) = Jacobian matrix of \( y \) at \( x \)

Example: \( f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3) \)

\[
y(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_2^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sim \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}
\]

Types

- \( n \in \mathbb{N} \) : input dimension
- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] \) : polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)

- monomial
Generable functions (generalization)

**Definition**

\( f : X \subseteq \mathbb{R}^n \to \mathbb{R} \) is generable if \( X \) is open connected and \( \exists d, p, x_0, y_0, y \) such that

\[
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**Types**

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- \( y_0 \in \mathbb{K}^d, y : X \to \mathbb{R}^d \)

**Example**

\( f(x_1, x_2) = x_1 x_2^2 \)

\[
\begin{align*}
y_1(0, 0) &= 0, & \partial_{x_1} y_1 &= y_3^2, & \partial_{x_2} y_1 &= 3y_2y_3 & \leadsto & & y_1(x) &= x_1 x_2^2 \\
y_2(0, 0) &= 0, & \partial_{x_1} y_2 &= 1, & \partial_{x_2} y_2 &= 0 & \leadsto & & y_2(x) &= x_1 \\
y_3(0, 0) &= 0, & \partial_{x_1} y_3 &= 0, & \partial_{x_2} y_3 &= 1 & \leadsto & & y_3(x) &= x_2
\end{align*}
\]

This is tedious!
Generable functions (generalization)

Definition

\[ f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \] is generable if \( X \) is open connected and \( \exists d, p, x_0, y_0, y \) such that

\[ y(x_0) = y_0, \quad J_y(x) = p(y(x)) \]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\[ J_y(x) \] = Jacobian matrix of \( y \) at \( x \)

Types

- \( n \in \mathbb{N} \) : input dimension
- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] \) : polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)
- inverse function

Last example : \( f(x) = \frac{1}{x} \) for \( x \in (0, \infty) \)

\[ y(1) = 1, \quad \partial_x y = -y^2 \quad \sim \quad y(x) = \frac{1}{x} \]
Generable functions: summary

Nice theory for the class of multivariate generable functions (over connected domains):

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$
- stable under $\pm$, $\times$, $/$, $\circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
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- technicality on the field \( K \) of coefficients for stability under \( \circ \)

Natural questions:

- analytic \( \rightarrow \) isn’t that very limited?
- can we generate all analytic functions?
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Nice theory for the class of multivariate generable functions (over connected domains):

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- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$

Natural questions:

- analytic $\rightarrow$ isn’t that very limited?
- can we generate all analytic functions? No

Riemann $\Gamma$ and $\zeta$ are not generable.
From discrete to real computability

**Computable Analysis**: lift Turing computability to real numbers

[Ko, 1991; Weihrauch, 2000]

Definition: $x \in \mathbb{R}$ is computable iff there exists a computable $f : \mathbb{N} \to \mathbb{Q}$ such that:

$$|x - f(n)| \leq 10^{-n}$$

$n \in \mathbb{N}$

Examples: rational numbers, $\pi$, $e$, ...

$$|\pi - f(n)| \leq 10^{-0} \leq 1$$

$$|3.140 - f(1)| \leq 10^{-1} \leq 0.1$$

$$|3.1400 - f(2)| \leq 10^{-2} \leq 0.01$$

$$|3.1415926535 - f(10)| \leq 10^{-10} \leq 0.0000000001$$

Beware: there exists uncomputable real numbers!
Computable Analysis: lift Turing computability to real numbers

**Definition**

$x \in \mathbb{R}$ is computable iff $\exists$ a computable $f : \mathbb{N} \rightarrow \mathbb{Q}$ such that:

$$|x - f(n)| \leq 10^{-n}, \quad n \in \mathbb{N}$$

**Examples**: rational numbers, $\pi$, $e$, ...

| $n$ | $f(n)$     | $|\pi - f(n)|$     |
|-----|------------|--------------------|
| 0   | 3          | $0.14 \leq 10^{-0}$|
| 1   | 3.1        | $0.04 \leq 10^{-1}$|
| 2   | 3.14       | $0.001 \leq 10^{-2}$|
| 10  | 3.1415926535 | $0.9 \cdot 10^{-10} \leq 10^{-10}$ |

Beware: there exists uncomputable real numbers!
Computable Analysis: lift Turing computability to real numbers

\[\text{Ko, 1991; Weihrauch, 2000}\]

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\(n \in \mathbb{N}\)

**Examples:** rational numbers, \(\pi\), \(e\), ...

| n  | \(f(n)\)     | \(|\pi - f(n)|\)         |
|----|--------------|--------------------------|
| 0  | 3            | 0.14 \(\leq 10^{-0}\)   |
| 1  | 3.1          | 0.04 \(\leq 10^{-1}\)   |
| 2  | 3.14         | 0.001 \(\leq 10^{-2}\)  |
| 10 | 3.1415926535 | 0.9 \(\cdot 10^{-10} \leq 10^{-10}\) |

**Beware:** there exists uncomputable real numbers!
From discrete to real computability

**Definition (Computable function)**

\( f : [a, b] \rightarrow \mathbb{R} \) is computable iff 

\[
\exists m : \mathbb{N} \rightarrow \mathbb{N}, \quad \psi : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q}
\]

computable functions such that:

\[
|x - y| \leq 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leq 10^{-n} \quad x, y \in \mathbb{R}, \quad n \in \mathbb{N}
\]

Polytime complexity

Add “polynomial time computable” everywhere.
From discrete to real computability

Definition (Computable function)

\( f : [a, b] \rightarrow \mathbb{R} \) is computable iff \( \exists m : \mathbb{N} \rightarrow \mathbb{N} \), computable functions such that:

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\]

\( m : \) modulus of continuity
Definition (Computable function)

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\]

\( m : \) modulus of continuity
From discrete to real computability

Definition (Computable function)

\[ f : [a, b] \to \mathbb{R} \text{ is computable iff } \exists \ m : \mathbb{N} \to \mathbb{N}, \ \psi : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \]

computable functions such that:

\[ |x - y| \leq 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leq 10^{-n} \quad x, y \in \mathbb{R}, n \in \mathbb{N} \]

\[ |f(r) - \psi(r, n)| \leq 10^{-n} \quad r \in \mathbb{Q}, n \in \mathbb{N} \]
Definition (Computable function)

\( f : [a, b] \rightarrow \mathbb{R} \) is computable iff \( \exists m : \mathbb{N} \rightarrow \mathbb{N}, \psi : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q} \)
computable functions such that:

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\[
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\]

\[
|f(r) - \psi(r, n)| \leq 10^{-n} \quad r \in \mathbb{Q}, n \in \mathbb{N}
\]

**Examples** : polynomials, \( \sin \), \( \exp \), \( \sqrt{\cdot} \).

**Note** : all computable functions are continuous

**Beware** : there exists (continuous) uncomputable real functions!
Definition (Computable function)

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. $f$ is computable iff there exist computable functions $m : \mathbb{N} \rightarrow \mathbb{N}$ and $\psi : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q}$ such that:

$$|x - y| \leq 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leq 10^{-n} \quad x, y \in \mathbb{R}, n \in \mathbb{N}$$

$$|f(r) - \psi(r, n)| \leq 10^{-n} \quad r \in \mathbb{Q}, n \in \mathbb{N}$$

Examples: polynomials, $\sin$, $\exp$, $\sqrt{\cdot}$

Note: all computable functions are continuous

Beware: there exists (continuous) uncomputable real functions!

Polytime complexity

Add “polynomial time computable” everywhere.
Equivalence with computable analysis

**Definition (Bournez et al, 2007)**

Let \( f \) be computable by GPAC if there exists a polynomial \( p \) such that for all \( x \in [a, b] \):

\[
\begin{align*}
  y(0) &= (x, 0, \ldots, 0) \\  y'(t) &= p(y(t))
\end{align*}
\]

satisfies \(|f(x) - y_1(t)| \leq y_2(t)\) and \(y_2(t) \to 0\) as \(t \to \infty\).

Theorem (Bournez et al, 2007)

\( f : [a, b] \to \mathbb{R} \) computable \( \iff \) \( f \) computable by GPAC.

1. In Computable Analysis, a standard model over reals built from Turing machines.
**Equivalence with computable analysis**

**Definition (Bournez et al, 2007)**

\[ f \text{ computable by GPAC if } \exists p \text{ polynomial such that } \forall x \in [a, b] \]

\[ y(0) = (x, 0, \ldots, 0) \quad y'(t) = p(y(t)) \]

satisfies \[ |f(x) - y_1(t)| \leq y_2(t) \text{ et } y_2(t) \xrightarrow{t \to \infty} 0. \]

\[ f(x) \xrightarrow{t \to \infty} f(x) \]

\[ y_2(t) = \text{error bound} \]

**Theorem (Bournez et al, 2007)**

\[ f : [a, b] \to \mathbb{R} \text{ computable}^1 \iff f \text{ computable by GPAC} \]
### Equivalence with computable analysis

**Definition (Bournez et al, 2007)**

A function $f$ is **computable by GPAC** if there exists a polynomial $p$ such that for all $x \in [a, b]$, the system of differential equations

$$
\begin{align*}
y(0) &= (x, 0, \ldots, 0) \\
y'(t) &= p(y(t))
\end{align*}
$$

satisfies $|f(x) - y_1(t)| \leq y_2(t)$ and $y_2(t) \to 0$ as $t \to \infty$.

### Theorem (Bournez et al, 2007)

$$
\begin{align*}
f : [a, b] \to \mathbb{R} & \text{ computable}^1 \iff f \text{ computable by GPAC}
\end{align*}
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---

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Almost-rounding function

“Perfect round”:

\[
\text{round}(x) := x - \frac{1}{\pi} \arctan(\tan(\pi x)).
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Almost-rounding function

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Undefined at \( x = n + \frac{1}{2} \): observe that

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\tan(\theta) = \text{sgn}(\theta) \frac{\sin(\theta)}{|\cos(\theta)|}
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Approximate \( \text{sgn}(\theta) \):

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\text{sgn}(\theta) \approx \tanh(\lambda x) \quad \text{for big } \lambda
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Prevent explosion:

$$|\cos(\theta)| \sim \sqrt{n z(\cos(\theta)^2)}$$

where $nz(x) \approx x$ but $nz(x) > 0$ for all $x$:
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Prevent explosion:

\[
|\cos(\theta)| \sim \sqrt{\text{nz}(\cos(\theta)^2)}
\]

where \( \text{nz}(x) \approx x \) but \( \text{nz}(x) > 0 \) for all \( x \):

\[
\text{nz}(x) = x + \text{some variation on } \tanh
\]
Almost-rounding function: gory details

Formally:

\[
\text{rnd}(x, \mu, \lambda) = x - \frac{1}{\pi} \arctan(\text{cltan}(\pi x, \mu, \lambda))
\]

\[
\text{cltan}(\theta, \mu, \lambda) = \frac{\sin(\theta)}{\sqrt{\text{nz}(\cos^2 \theta, \mu + 16\lambda^3, 4\lambda^2)}} \quad \text{sg}(\cos \theta, \mu + 3\lambda, 2\lambda)
\]

\[
\text{nz}(x, \mu, \lambda) = x + \frac{2}{\lambda} \text{ip}_1 \left(1 - x + \frac{3}{4\lambda}, \mu + 1, 4\lambda\right)
\]

\[
\text{ip}_1(x, \mu, \lambda) = \frac{1 + \text{sg}(x - 1, \mu, \lambda)}{2}
\]

\[
\text{sg}(x, \mu, \lambda) = \tanh(x\mu\lambda)
\]

All generable functions!