Continuous models of computation: computability, complexity, universality

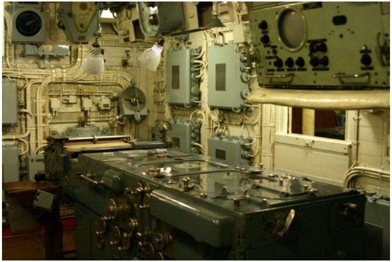
Amaury Pouly

21 mars 2017

Digital vs analog computers

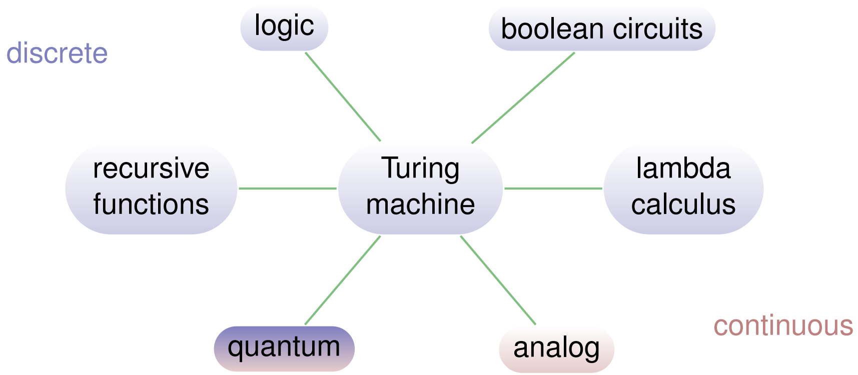
Digital vs analog computers

VS



Church Thesis

Computability

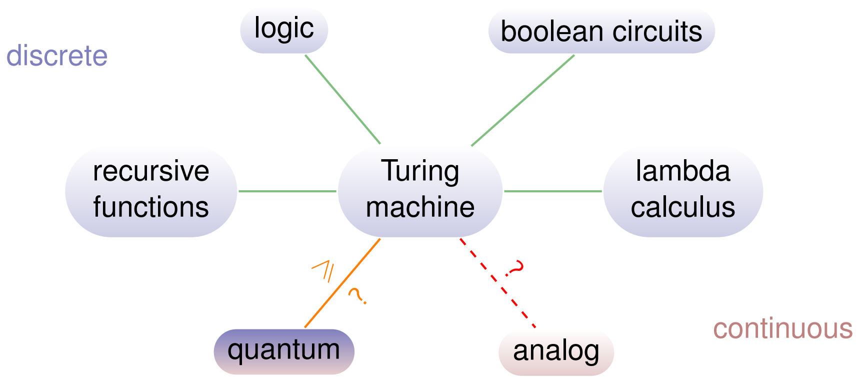


Church Thesis

All reasonable models of computation are equivalent.

Church Thesis

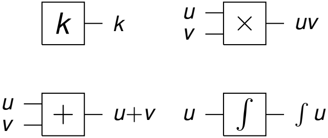
Complexity



Effective Church Thesis

All reasonable models of computation are equivalent for complexity.

Polynomial Differential Equations



General Purpose
Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks :

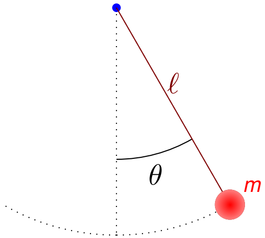
- chemical
- enzymatic

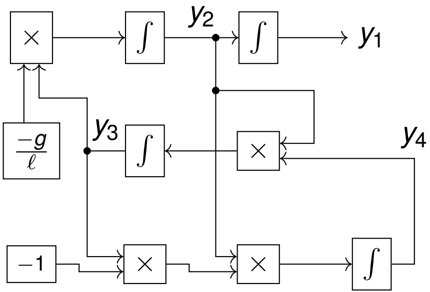
polynomial differential equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable $(+, \times, \circ, /, \mathrm{ED})$
- No closed-form solution

Example of dynamical system





$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

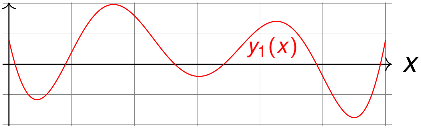
$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{l} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

Computing with the GPAC

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



Shannon's notion

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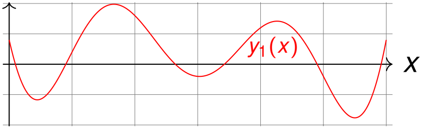

Shannon's notion

sin, cos, exp, log, ...

Strictly weaker than Turing machines [Shannon, 1941]

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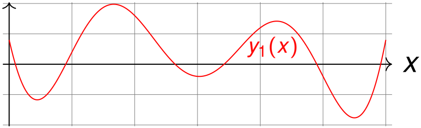

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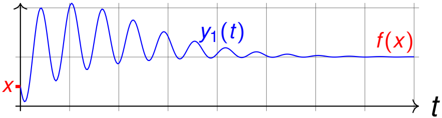
Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$



Modern notion

Computing with the GPAC

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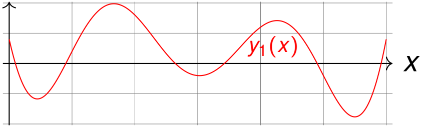

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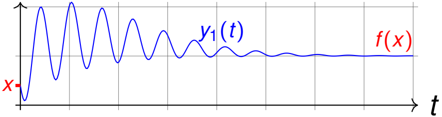

Modern notion

sin, cos, exp, log, Γ, ζ, ...

Turing powerful [Bournez et al., 2007]

From discrete to real computability

Computable Analysis : lift Turing computability to real numbers
[Ko, 1991 ; Weihrauch, 2000]

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Definition

$x \in \mathbb{R}$ is computable iff $\exists$ a computable $f : \mathbb{N} \to \mathbb{Q}$ such that :

$$|x - f(n)| \leqslant 10^{-n} \qquad n \in \mathbb{N}$$

Examples : rational numbers, π, e, ...

n	**f(n)**	$\lvert\pi - \mathbf{f}(\mathbf{n})\rvert$
0	3	$0.14 \leqslant 10^{-0}$
1	3.1	$0.04 \leqslant 10^{-1}$
2	3.14	$0.001 \leqslant 10^{-2}$
10	3.1415926535	$0.9 \cdot 10^{-10} \leqslant 10^{-10}$

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Beware : there exists uncomputable real numbers !

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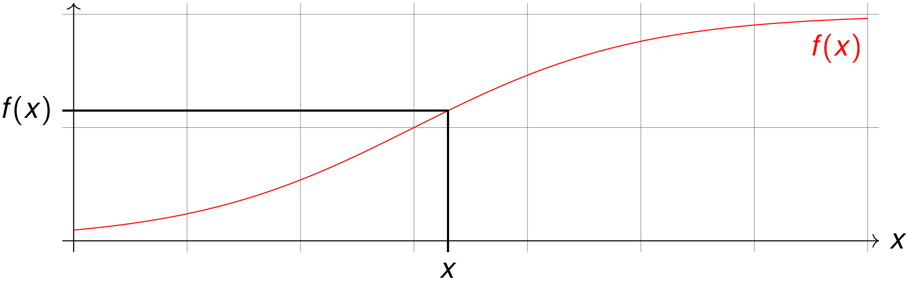

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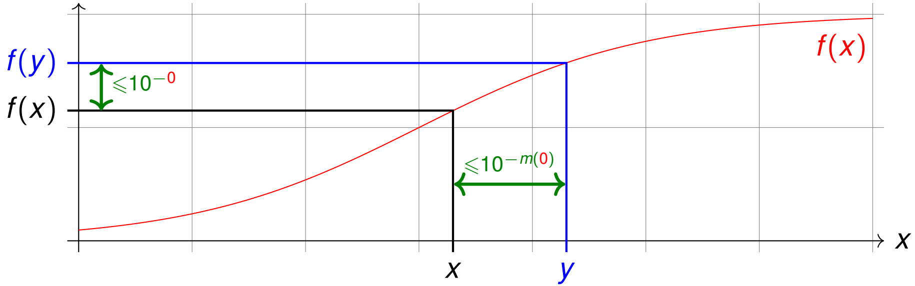

Definition (Computable function)

$f : [a, b] \to \mathbb{R}$ is computable iff $\exists\, m : \mathbb{N} \to \mathbb{N}$, computable functions such that :

$$|x - y| \leqslant 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leqslant 10^{-n} \qquad x, y \in \mathbb{R}, n \in \mathbb{N}$$

m : modulus of continuity

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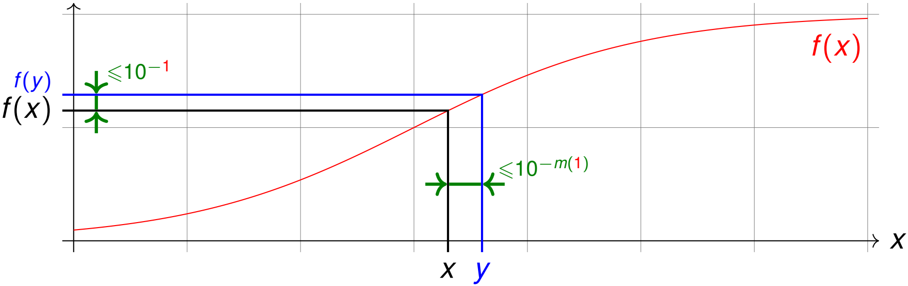

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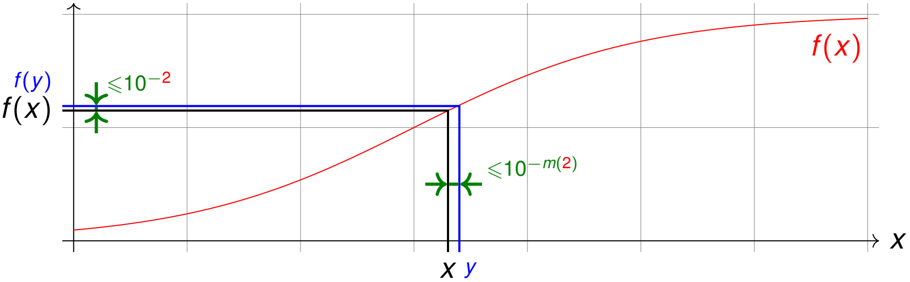

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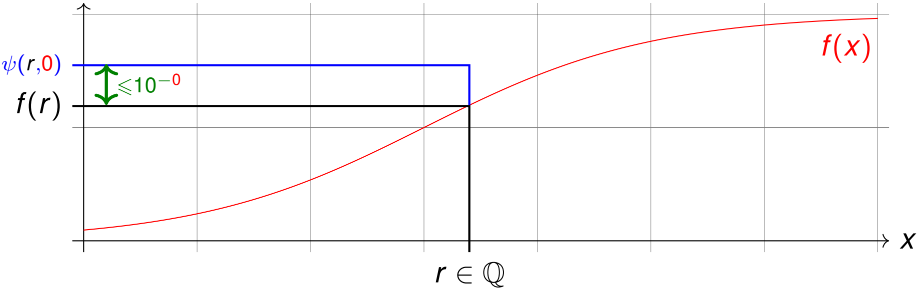

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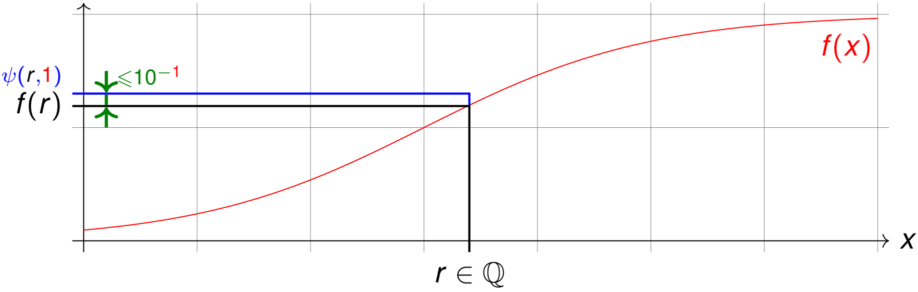

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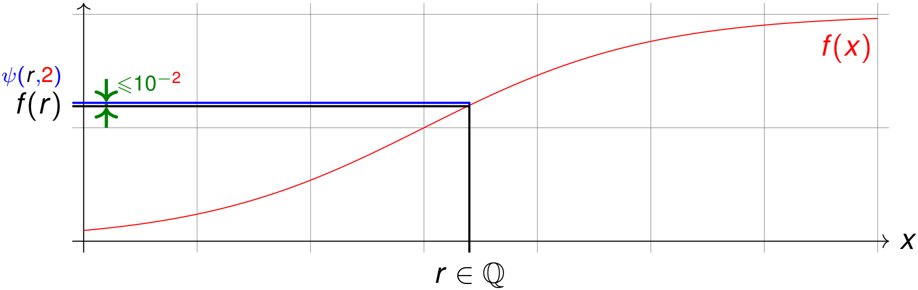

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Examples : polynomials, $\sin$, $\exp$, $\sqrt{\cdot}$

Note : all computable functions are continuous

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Polytime complexity

Add "polynomial time computable" everywhere.

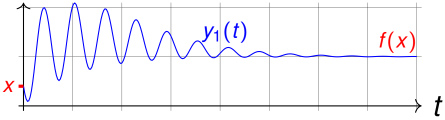
Equivalence with computable analysis

Definition (Bournez et al)

f computable by GPAC if $\exists p$ polynomial such that $\forall x$

$$y(0) = (x, 0, \ldots, 0) \qquad y'(t) = p(y(t))$$

satisfies $|f(x) - y_1(t)| \leqslant y_2(t)$ et $y_2(t) \xrightarrow[t\to\infty]{} 0$.



$$y_1(t) \xrightarrow[t\to\infty]{} f(x)$$

$y_2(t) =$ error bound

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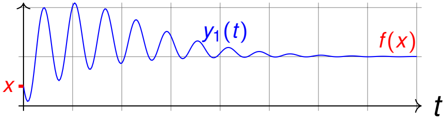

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Theorem (Bournez, Campagnolo, Graça, Hainry)

$f : [a, b] \to \mathbb{R}$ computable $\Leftrightarrow$ f computable by GPAC

Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x

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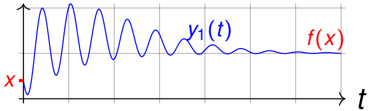
- GPAC : time contraction problem

Tentative definition

$T(x,\mu) =$ first time t so that $|y_1(t) - f(x)| \leqslant e^{-\mu}$

$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$



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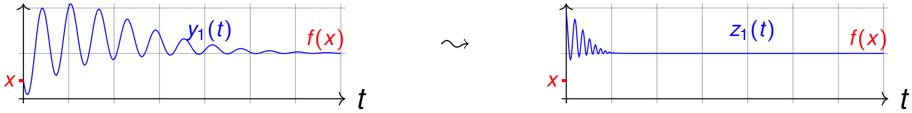

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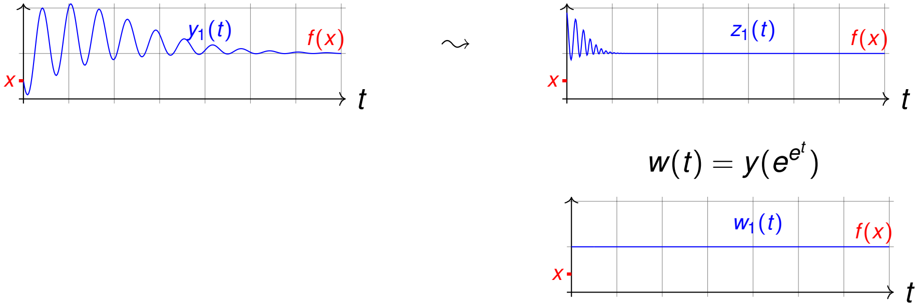

Complexity of analog systems

- Turing machines : $T(x)$ = number of steps to compute on x
- GPAC : time contraction problem $\rightarrow$ **open problem**

Tentative definition

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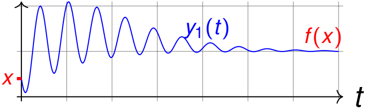
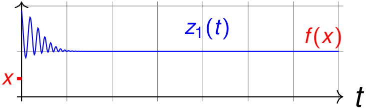


$\rightsquigarrow$

$z(t) = y(e^t)$



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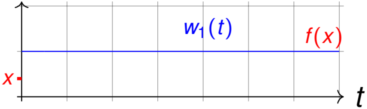
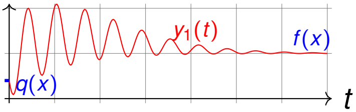


Problem

All functions have constant time complexity.

Time-space correlation of the GPAC

$y(0) = q(x) \qquad y' = p(y)$



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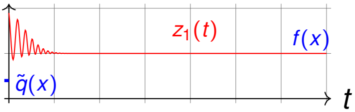

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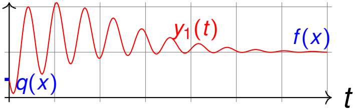

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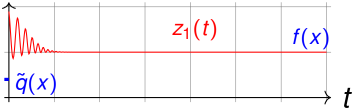
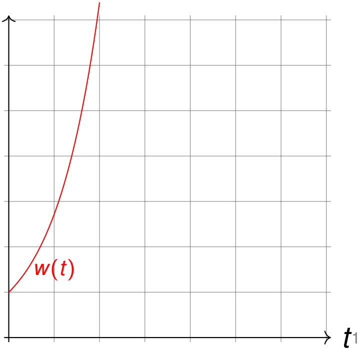


extra component : $w(t) = e^t$



Time-space correlation of the GPAC

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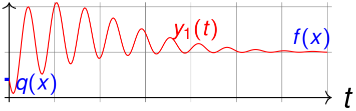

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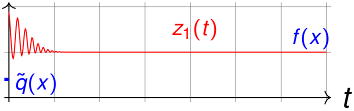
$z(t) = y(e^t)$



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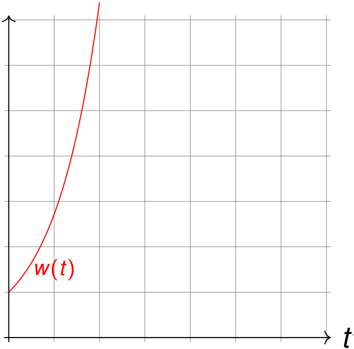


Observation

Time scaling costs “space”.

$\rightsquigarrow$

Time complexity for the GPAC must involve time and space !

Complexity of solving polynomial ODEs

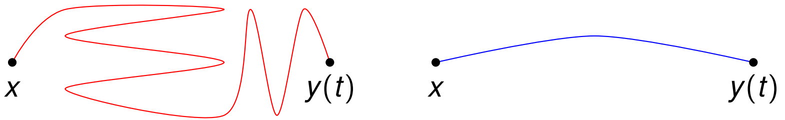
$$y(0) = x \qquad y'(t) = p(y(t))$$

Theorem (Graça, Pouly) [TCS 2016]

If $y(t)$ exists, one can compute p, q such that $\left|\frac{p}{q} - y(t)\right| \leqslant 2^{-n}$ in time

$$\operatorname{poly}(\texttt{size of } x \texttt{ and } p, n, \ell(t))$$

where $\ell(t) = \int_0^t \max(1, \|y(u)\|)^{\deg(p)} du \approx$ length of the curve

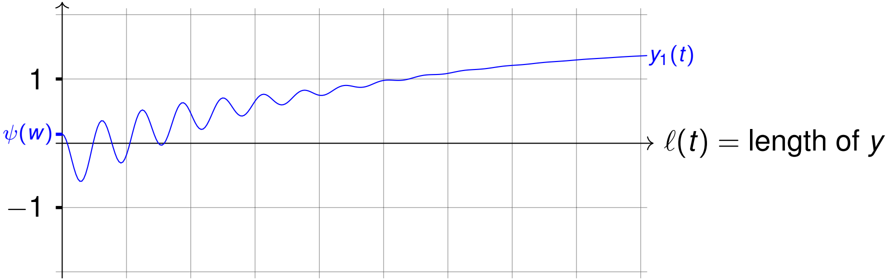


length of the curve = complexity = ressource

Characterization of polynomial time

Definition : $\mathcal{L} \in$ ANALOG-PTIME $\Leftrightarrow \exists p$ polynomial, $\forall$ word w

$$y(0) = (\psi(w), |w|, 0, \ldots, 0) \qquad y' = p(y) \qquad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}$$



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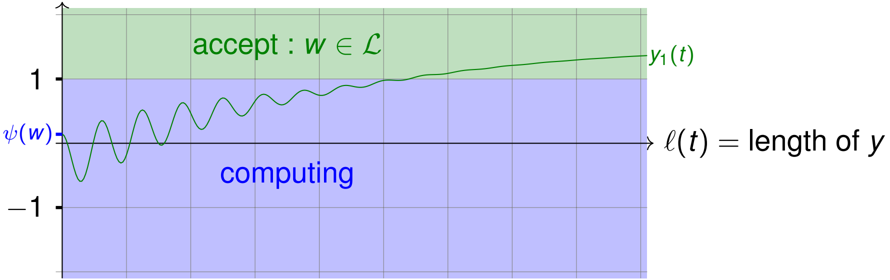

satisfies

1. if $y_1(t) \geqslant 1$ then $w \in \mathcal{L}$

Characterization of polynomial time

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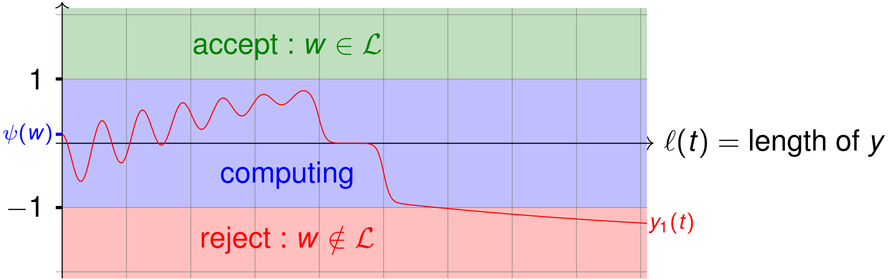

satisfies

2. if $y_1(t) \leqslant -1$ then $w \notin \mathcal{L}$

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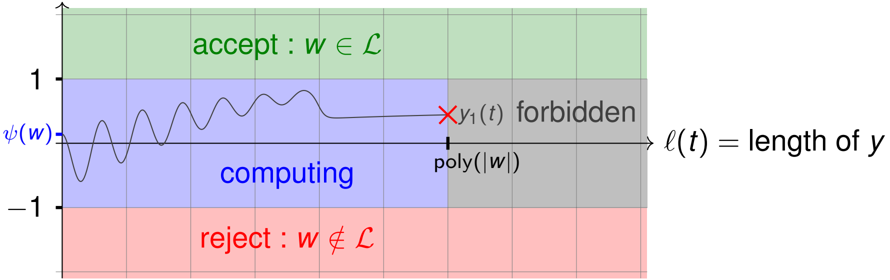

satisfies

3. if $\ell(t) \geqslant \text{poly}(|w|)$ then $|y_1(t)| \geqslant 1$

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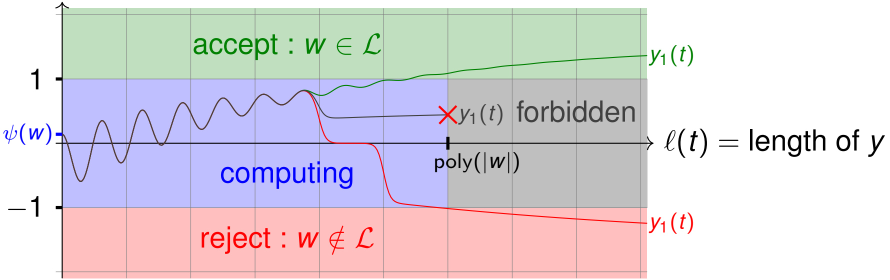

Theorem (JoC 2016 ; ICALP 2016)

PTIME = ANALOG-PTIME

Characterization of real polynomial time

Definition : $f : [a,b] \to \mathbb{R}$ in ANALOG-P$_{\mathbb{R}}$ $\Leftrightarrow$ $\exists p$ polynomial, $\forall x \in [a,b]$

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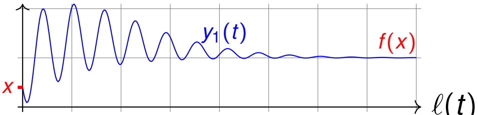

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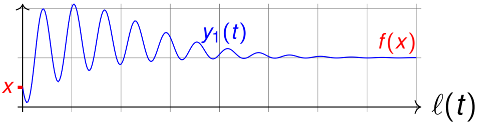
satisfies :

1. $|y_1(t) - f(x)| \leqslant 2^{-\ell(t)}$

   «greater length $\Rightarrow$ greater precision»

2. $\ell(t) \geqslant t$

   «length increases with time»



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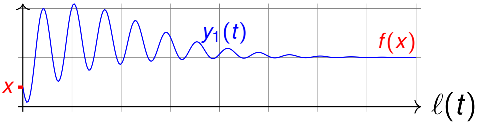
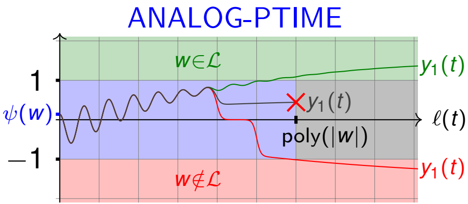
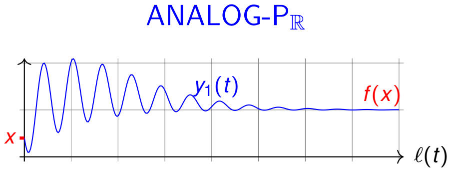


Theorem [JoC 2016 ; ICALP 2016]

$f : [a,b] \to \mathbb{R}$ computable in polynomial time $\Leftrightarrow$ $f \in$ ANALOG-P$_\mathbb{R}$.

Summary





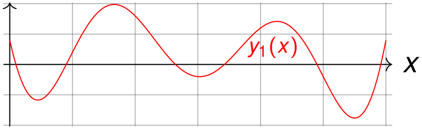
Theorem [JoC 2016 ; ICALP 2016]

- $\mathcal{L} \in$ PTIME of and only if $\mathcal{L} \in$ ANALOG-PTIME
- $f : [a, b] \to \mathbb{R}$ computable in polynomial time $\Leftrightarrow$ $f \in$ ANALOG-P$_\mathbb{R}$

- Analog complexity theory based on length
- time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME

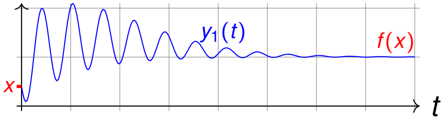
Universal differential equations

Generable functions



subclass of analytic functions

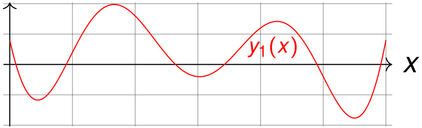
Computable functions



any computable function

Universal differential equations

Generable functions



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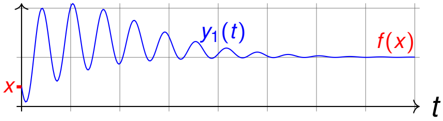
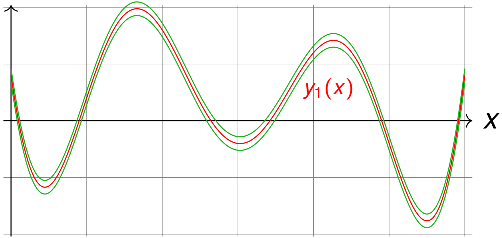
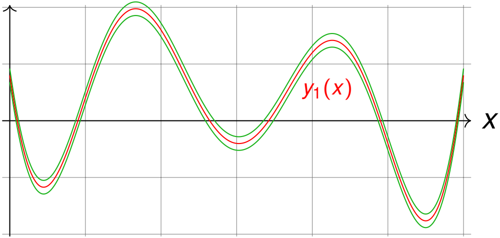
Computable functions



any computable function



Universal differential equation (Rubel)



Theorem (Rubel)

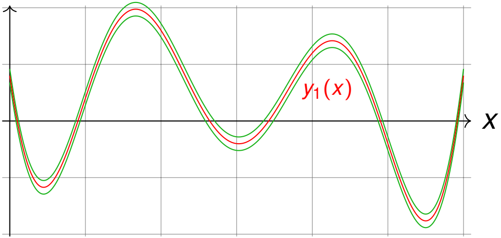
There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists a solution y to

$$p(t, y, y', \dots, y^{(k)}) = 0$$

such that

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Universal differential equation (Rubel)



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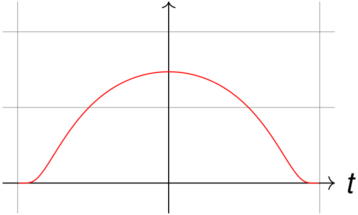
Problem : Rubel is «cheating».

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise

$$(1 - t^2)^2 f''(t) + 2tf'(t) = 0.$$



Rubel's proof in one slide

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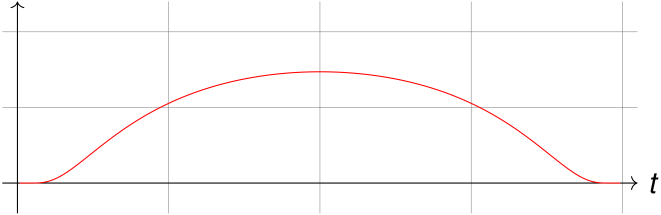
- Can do the same with $cf(at+b)$ (translation+scaling)



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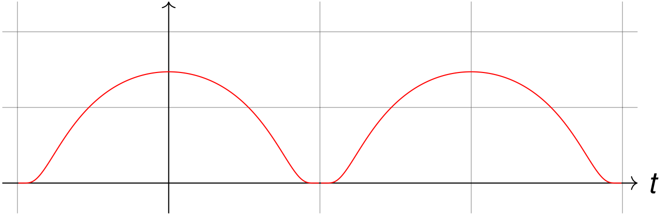
- Can glue together arbitrary many such pieces



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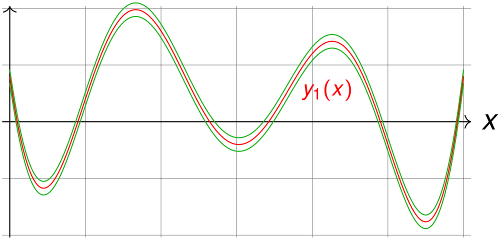
- Can do the same with $cf(at+b)$ (translation+scaling)
- Can glue together arbitrary many such pieces
- Can arrange so that $\int f$ is solution : almost piecewise linear

Universal differential equation (Rubel)



Theorem

There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** and this solution satisfies such that

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Future work

$\xleftrightarrow{?}$

Reaction networks :

- chemical
- enzymatic

$\updownarrow$

$$y' = p(y)$$

$\updownarrow$?

$$y' = p(y) + e(t)$$

- Finer time complexity (linear)
- Nondeterminism
- Robustness
- « space» complexity
- Other models
- Stochastic