Continuous models of computation: computability, complexity, universality

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Digital vs analog computers
Digital vs analog computers
Church Thesis

All reasonable models of computation are equivalent.
Effective Church Thesis

All *reasonable* models of computation are equivalent for complexity.
Polynomial Differential Equations

General Purpose Analog Computer

Newton mechanics

Reaction networks:
- chemical
- enzymatic

Differential Analyzer

polynomial differential equations:
\[ \begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases} \]

- Rich class
- Stable (+, \times, \circ, /, ED)
- No closed-form solution
Example of dynamical system

\[ \ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \]

\[
\begin{align*}
y_1' &= y_2 \\
y_2' &= -\frac{g}{\ell} y_3 \\
y_3' &= y_2 y_4 \\
y_4' &= -y_2 y_3
\end{align*}
\]

\[
\begin{align*}
y_1 &= \theta \\
y_2 &= \dot{\theta} \\
y_3 &= \sin(\theta) \\
y_4 &= \cos(\theta)
\end{align*}
\]
Generable functions

\[
\begin{cases}
y(0) = y_0 \\
y'(x) = p(y(x))
\end{cases}
\quad x \in \mathbb{R}
\]

\[f(x) = y_1(x)\]

Shannon’s notion
Generable functions

\[
\begin{align*}
  y(0) &= y_0 \\
  y'(x) &= p(y(x)) \\
  x &\in \mathbb{R}
\end{align*}
\]

\[f(x) = y_1(x)\]

Shannon’s notion

\[
\sin, \cos, \exp, \log, \ldots
\]

Strictly weaker than Turing machines [Shannon, 1941]
Computing with the GPAC

Generable functions

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Shannon’s notion

\[ \sin, \cos, \exp, \log, \ldots \]

Strictly weaker than Turing machines [Shannon, 1941]

Computable

\[
\begin{align*}
  y(0) &= q(x) \\
  y'(t) &= p(y(t))
\end{align*}
\]

\[ f(x) = \lim_{t \to \infty} y_1(t) \]

Modern notion

\[ \sin, \cos, \exp, \log, \ldots \]

Turing powerful [Bournez et al., 2007]
Computing with the GPAC

Generable functions
\[
\begin{align*}
    y(0) &= y_0 \\
y'(x) &= p(y(x))
\end{align*}
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\[x \in \mathbb{R}\]

\[f(x) = y_1(x)\]

Shannon’s notion
sin, cos, exp, log, ...

Strictly weaker than Turing machines [Shannon, 1941]

Computable
\[
\begin{align*}
    y(0) &= q(x) \\
y'(t) &= p(y(t))
\end{align*}\]
\[x \in \mathbb{R}\]
\[t \in \mathbb{R}_+\]

\[f(x) = \lim_{t \to \infty} y_1(t)\]

Modern notion
sin, cos, exp, log, \(\Gamma, \zeta, \ldots\)

Turing powerful
[Bournez et al., 2007]
From discrete to real computability

**Computable Analysis**: lift Turing computability to real numbers

[Ko, 1991; Weihrauch, 2000]

\[
\text{x} \in \mathbb{R} \text{ is computable iff } \exists \text{ a computable } f : \mathbb{N} \rightarrow \mathbb{Q} \text{ such that:}
\]

\[
|\text{x} - f(n)| \leq 10^{-n} \quad n \in \mathbb{N}
\]

Examples: rational numbers, \(\pi\), \(e\), ...

\[
|\pi - f(n)| \leq 10^{-0} = 0.14 \leq 10^{-0} = 1
\]

\[
|\pi - f(n)| \leq 10^{-1} = 0.04 \leq 10^{-1} = 1
\]

\[
|\pi - f(n)| \leq 10^{-2} = 0.001 \leq 10^{-2} = 1
\]

\[
|\pi - f(n)| \leq 10^{-10} = 10^{-10} \leq 10^{-10} = 1
\]

Beware: there exists uncomputable real numbers!
Computable Analysis: lift Turing computability to real numbers

[Ко, 1991; Вейхрах, 2000]

**Definition**

$x \in \mathbb{R}$ is computable iff $\exists$ a computable $f : \mathbb{N} \to \mathbb{Q}$ such that:

$$|x - f(n)| \leq 10^{-n} \quad n \in \mathbb{N}$$

**Examples:** rational numbers, $\pi$, $e$, ...

| $n$ | $f(n)$ | $|\pi - f(n)|$ |
|-----|--------|----------------|
| 0   | 3      | $0.14 \leq 10^{-0}$ |
| 1   | 3.1    | $0.04 \leq 10^{-1}$ |
| 2   | 3.14   | $0.001 \leq 10^{-2}$ |
| 10  | 3.1415926535 | $0.9 \cdot 10^{-10} \leq 10^{-10}$ |

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Computable Analysis: lift Turing computability to real numbers

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**Beware:** there exists uncomputable real numbers!
Definition (Computable function)

\( f : [a, b] \rightarrow \mathbb{R} \) is computable iff

\[ \exists m : \mathbb{N} \rightarrow \mathbb{N}, \psi : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q} \]

such that:

\[ |x - y| \leq 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leq 10^{-n} \]

\( x, y \in \mathbb{R}, n \in \mathbb{N} \)

Polytime complexity

Add "polynomial time computable" everywhere.
Definition (Computable function)

\[ f : [a, b] \rightarrow \mathbb{R} \] is computable iff \( \exists m : \mathbb{N} \rightarrow \mathbb{N}, \) computable functions such that:

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\( m \) : modulus of continuity
Definition (Computable function)

\[ f : [a, b] \rightarrow \mathbb{R} \text{ is computable iff } \exists m : \mathbb{N} \rightarrow \mathbb{N}, \psi : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q} \]

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\( m : \) modulus of continuity
From discrete to real computability

\[ f(x) \]

\[ r \in \mathbb{Q} \]

**Definition (Computable function)**

\[ f : [a, b] \rightarrow \mathbb{R} \text{ is computable iff } \exists m : \mathbb{N} \rightarrow \mathbb{N}, \psi : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q} \]

computable functions such that:

\[ |x - y| \leq 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leq 10^{-n} \quad x, y \in \mathbb{R}, n \in \mathbb{N} \]

\[ |f(r) - \psi(r, n)| \leq 10^{-n} \quad r \in \mathbb{Q}, n \in \mathbb{N} \]
From discrete to real computability

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Examples: polynomials, \( \sin \), \( \exp \), \( \sqrt{\cdot} \).

Note: all computable functions are continuous

Beware: there exists (continuous) uncomputable real functions!
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Note: all computable functions are continuous.

Beware: there exists (continuous) uncomputable real functions!

Polytime complexity

Add “polynomial time computable” everywhere.
Equivalence with computable analysis

**Definition (Bournez et al)**

\( f \) computable by GPAC if \( \exists p \) polynomial such that \( \forall x \)

\[
y(0) = (x, 0, \ldots, 0) \quad y'(t) = p(y(t))
\]

satisfies \( |f(x) - y_1(t)| \leq y_2(t) \text{ and } y_2(t) \xrightarrow{t \to \infty} 0. \)

**Theorem (Bournez, Campagnolo, Graça, Hainry)**

\( f : [a, b] \to \mathbb{R} \) computable \( \iff \) \( f \) computable by GPAC

\[ y_1(t) \xrightarrow{t \to \infty} f(x) \]

\[ y_2(t) = \text{error bound} \]
Equivalence with computable analysis

**Definition (Bournez et al)**

$f$ computable by GPAC if $\exists p$ polynomial such that $\forall x$

\[ y(0) = (x, 0, \ldots, 0) \quad y'(t) = p(y(t)) \]

satisfies $|f(x) - y_1(t)| \leq y_2(t)$ et $y_2(t) \xrightarrow{t \to \infty} 0$.

**Theorem (Bournez, Campagnolo, Graça, Hainry)**

$f : [a, b] \to \mathbb{R}$ computable $\iff$ $f$ computable by GPAC
Turing machines: $T(x) =$ number of steps to compute on $x$
Complexity of analog systems

- Turing machines: $T(x) =$ number of steps to compute on $x$
- GPAC: time contraction problem

**Tentative definition**

$$T(x, \mu) = \text{first time } t \text{ so that } |y_1(t) - f(x)| \leq e^{-\mu}$$

$y(0) = (x, 0, \ldots, 0) \quad y' = p(y)$
Complexity of analog systems

- **Turing machines**: $T(x) =$ number of steps to compute on $x$
- **GPAC**: time contraction problem

**Tentative definition**

$$T(x, \mu) = \text{first time } t \text{ so that } |y_1(t) - f(x)| \leq e^{-\mu}$$

$y(0) = (x, 0, \ldots, 0)$  $y' = p(y)$  $z(t) = y(e^t)$

![Graph showing $y_1(t)$, $f(x)$, and $z_1(t)$](image)
Complexity of analog systems

- **Turing machines**: \( T(x) \) = number of steps to compute on \( x \)
- **GPAC**: time contraction problem

### Tentative definition

\[
T(x, \mu) = \text{first time } t \text{ so that } |y_1(t) - f(x)| \leq e^{-\mu}
\]

\[
y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \quad \quad z(t) = y(e^t)
\]

\[
w(t) = y(e^{e^t})
\]

\[
z_1(t) \quad f(x)
\]

\[
w_1(t) \quad f(x)
\]
Complexity of analog systems

- Turing machines: $T(x) =$ number of steps to compute on $x$
- GPAC: time contraction problem → open problem

**Tentative definition**

$T(x, \mu) =$ first time $t$ so that $|y_1(t) - f(x)| \leq e^{-\mu}$

$y(0) = (x, 0, \ldots, 0)$ \quad $y' = p(y)$

$z(t) = y(e^t)$

$w(t) = y(e^{e^t})$

**Problem**

All functions have constant time complexity.
Time-space correlation of the GPAC

\[ y(0) = q(x) \quad y' = p(y) \]

\[ z(t) = y(e^t) \]

**Observation**
Time scaling costs "space".

Time complexity for the GPAC must involve time and space!
Time-space correlation of the GPAC

\[ y(0) = q(x) \quad y' = p(y) \]

\[ z(t) = y(e^t) \]

Observation: Time scaling costs "space".

Time complexity for the GPAC must involve time and space!

Extra component: \( w(t) = e^t \)
Time-space correlation of the GPAC

\[ y(0) = q(x) \quad y' = p(y) \]

\[ z(t) = y(e^t) \]

Observation
Time scaling costs “space”.

Time complexity for the GPAC must involve time and space!
Complexity of solving polynomial ODEs

\[ y(0) = x \quad y'(t) = p(y(t)) \]

**Theorem (Graça, Pouly) [TCS 2016]**

If \( y(t) \) exists, one can compute \( p, q \) such that \( \left| \frac{p}{q} - y(t) \right| \leq 2^{-n} \) in time

\[
\text{poly}( \text{size of } x \text{ and } p, n, \ell(t) )
\]

where \( \ell(t) = \int_0^t \max(1, \|y(u)\|)^{\deg(p)} \, du \approx \text{length of the curve} \)

\[ x \quad y(t) \quad x \quad y(t) \]

length of the curve = complexity = ressource
**Definition**: $\mathcal{L} \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial, } \forall \text{ word } w$

$$y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}$$

$\ell(t) = \text{length of } y$
Characterization of polynomial time

**Definition**: \( \mathcal{L} \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial}, \forall \text{ word } w \)

\[
y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{|w|} w_i2^{-i}
\]

satisfies

1. if \( y_1(t) \geq 1 \) then \( w \in \mathcal{L} \)
Characterization of polynomial time

**Definition:** \( \mathcal{L} \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial}, \ \forall \text{ word } w \)

\[
y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{\frac{|w|}{12}} w_i 2^{-i}
\]

- **accept**: \( w \in \mathcal{L} \)
- **reject**: \( w \notin \mathcal{L} \)

satisfies

1. if \( y_1(t) \leq -1 \) then \( w \notin \mathcal{L} \)
Characterization of polynomial time

**Definition:** \( L \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial}, \forall \text{ word } w \)

\[
y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}
\]

- **accept:** \( w \in L \)
- **reject:** \( w \notin L \)

satisfies
- if \( \ell(t) \geq \text{poly}(|w|) \) then \( |y_1(t)| \geq 1 \)
Characterization of polynomial time

**Definition:** $\mathcal{L} \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial, } \forall \text{ word } w$

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y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{|w|} w_i2^{-i}
\]

**Theorem (JoC 2016; ICALP 2016)**

$\text{PTIME} = \text{ANALOG-PTIME}$
Characterization of real polynomial time

**Definition:** \( f : [a, b] \rightarrow \mathbb{R} \) in \( \text{ANALOG-P}_{\mathbb{R}} \) \( \iff \exists p \) polynomial, \( \forall x \in [a, b] \)

\[
y(0) = (x, 0, \ldots, 0) \quad y' = p(y)
\]
Characterization of real polynomial time

**Definition:** \( f : [a, b] \rightarrow \mathbb{R} \) in ANALOG-\( P_{\mathbb{R}} \) \iff \exists p \text{ polynomial}, \ \forall x \in [a, b]

\[
y(0) = (x, 0, \ldots, 0) \quad y' = p(y)
\]
satisfies:

1. \(|y_1(t) - f(x)| \leq 2^{-\ell(t)}

   «greater length \implies greater precision»

2. \(\ell(t) \geq t\)

   «length increases with time»

![Graph showing the relationship between \(f(x)\) and \(y_1(t)\) over time]
Characterization of real polynomial time

**Definition:** \( f : [a, b] \to \mathbb{R} \) in \( \text{ANALOG-P}_{\mathbb{R}} \iff \exists p \text{ polynomial}, \forall x \in [a, b] \\
y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \\
\) satisfies:

1. \(|y_1(t) - f(x)| \leq 2^{-\ell(t)}
\) «greater length \( \Rightarrow \) greater precision»

2. \(\ell(t) \geq t\)
\) «length increases with time»

**Theorem** [JoC 2016; ICALP 2016]

\( f : [a, b] \to \mathbb{R} \) computable in polynomial time \( \iff \) \( f \in \text{ANALOG-P}_{\mathbb{R}} \).
Theorem [JoC 2016; ICALP 2016]

- $\mathcal{L} \in \text{PTIME}$ of and only if $\mathcal{L} \in \text{ANALOG-PTIME}$
- $f : [a, b] \to \mathbb{R}$ computable in polynomial time $\iff f \in \text{ANALOG-P}_\mathbb{R}$

- Analog complexity theory based on length
- time of Turing machine $\iff$ length of the GPAC
- Purely continuous characterization of PTIME
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Theorem (Rubel)

There exists a fixed polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists a solution $y$ to

$$p(t, y, y', \ldots, y^{(k)}) = 0$$

such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Theorem (Rubel)

There exists a \textbf{fixed} polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists a solution $y$ to

$$p(t, y, y', \ldots, y^{(k)}) = 0$$

such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$

Problem: Rubel is «cheating». 
Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

$$(1 - t^2)^2 f''(t) + 2tf'(t) = 0.$$
Rubel’s proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise
  
  $$(1 - t^2)^2 f''(t) + 2tf'(t) = 0.$$  

- Can do the same with $cf(at + b)$ (translation+scaling)
Rubel’s proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.
  
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- Can do the same with $cf(at + b)$ (translation+scaling).
- Can glue together arbitrary many such pieces.
Rubel’s proof in one slide

- Take \( f(t) = e^{\frac{1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise

\[
(1 - t^2)^2 f''(t) + 2tf'(t) = 0.
\]

- Can do the same with \( cf(at + b) \) (translation+scaling)
- Can glue together arbitrary many such pieces
- Can arrange so that \( \int f \) is solution: almost piecewise linear
There exists a fixed polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

has a unique analytic solution and this solution satisfies such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Future work

Reaction networks:
- chemical
- enzymatic

\[ y' = p(y) \]
\[ y' = p(y) + e(t) \]

- Finer time complexity (linear)
- Nondeterminism
- Robustness
- « space» complexity
- Other models
- Stochastic