Continuous models of computation: computability, complexity, universality

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Joint work with Olivier Bournez and Daniel Graça

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Teaser

Characterization of P using differential equations

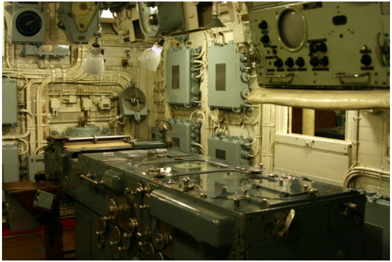
Universal differential equation

What is a computer?

What is a computer ?

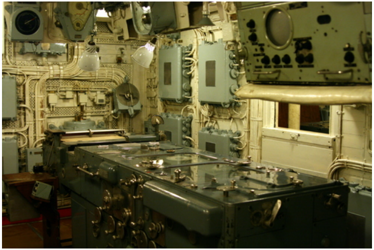
What is a computer ?

VS

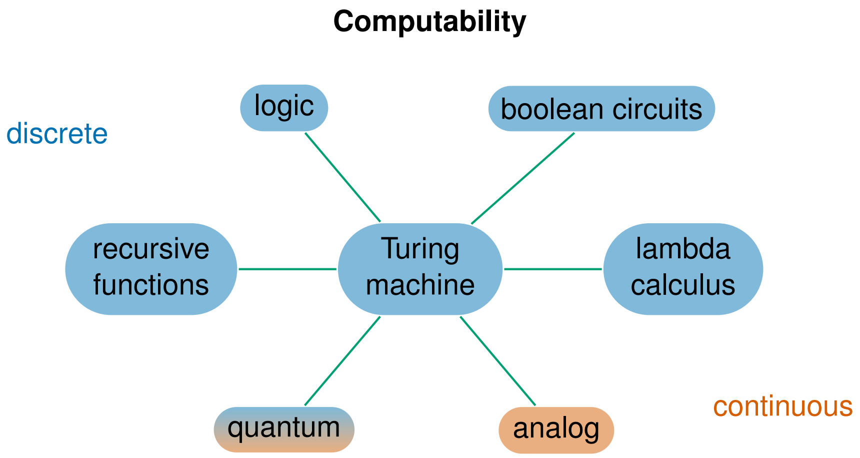


Analog Computers

Differential Analyser
“Mathematica of the 1920s”



Admiralty Fire Control Table
British Navy ships (WW2)

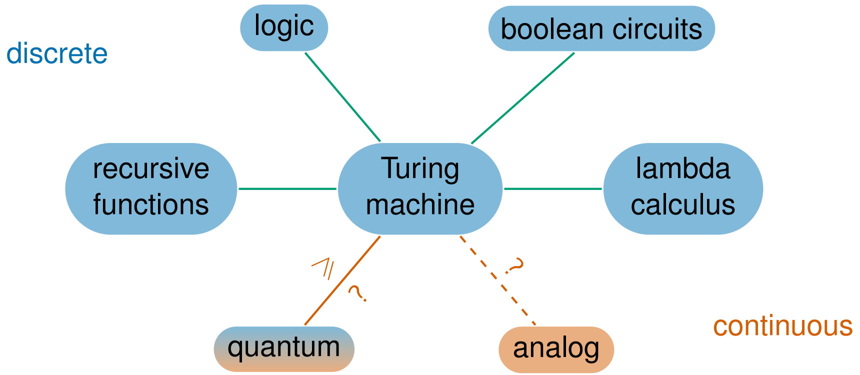


Church Thesis

All **reasonable** models of computation are equivalent.

Church Thesis

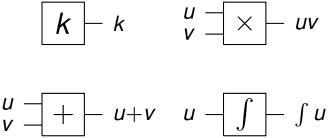
Complexity



Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.

Polynomial Differential Equations



General Purpose
Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks :

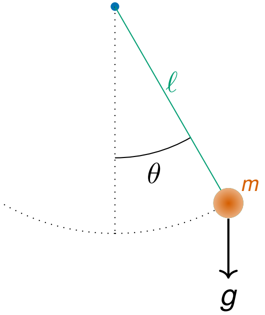
- chemical
- enzymatic

polynomial differential equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable ($+, \times, \circ, /$, ED)
- No closed-form solution

Example of dynamical system



$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

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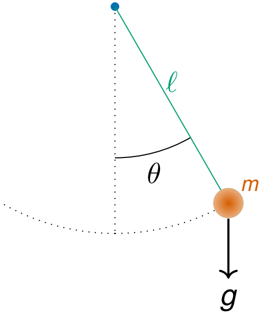

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$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{l} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

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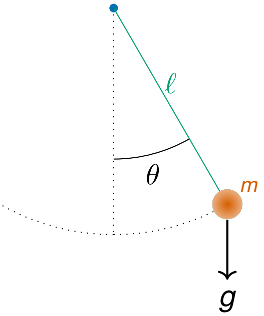

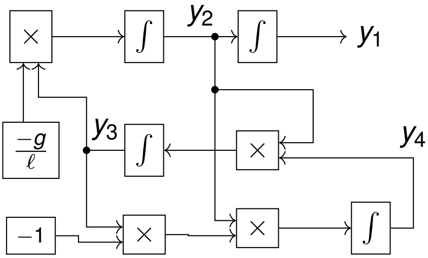

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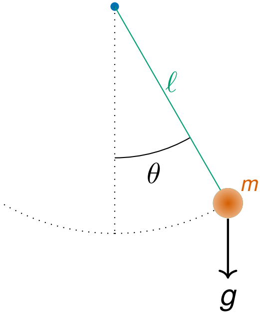

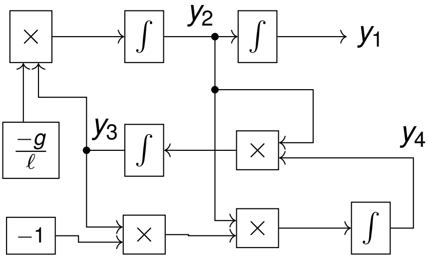

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Historical remark : the word “analog”

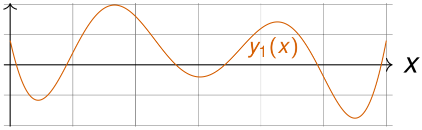
The pendulum and the circuit have the same equation. One can study one using the other by analogy.

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$f(x) = y_1(x)$



Shannon's notion

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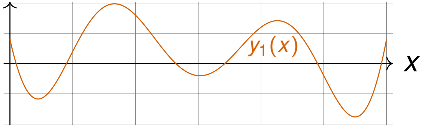

Shannon's notion

$\sin, \cos, \exp, \log, \ldots$

Strictly weaker than Turing machines [Shannon, 1941]

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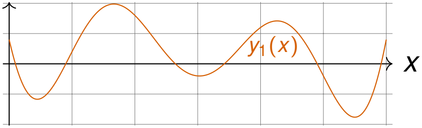

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sin, cos, exp, log, ...

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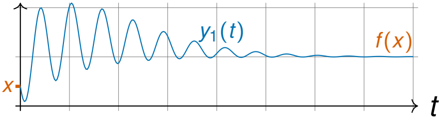
Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t\to\infty} y_1(t)$$



Modern notion

Computing with differential equations

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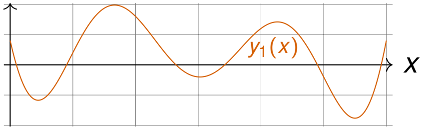

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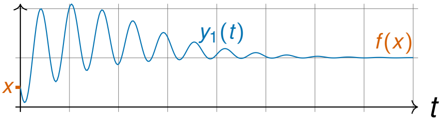

Modern notion

sin, cos, exp, log, Γ, ζ, ...

Turing powerful [Bournez et al., 2007]

From discrete to real computability

Computable Analysis : “Turing” computability over real numbers

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Definition (Ko, 1991 ; Weihrauch, 2000)

$x \in \mathbb{R}$ is computable iff $\exists$ a computable $f : \mathbb{N} \to \mathbb{Q}$ such that :

$$|x - f(n)| \leqslant 10^{-n} \qquad n \in \mathbb{N}$$

Examples : rational numbers, π, e, ...

$\mathbf{n}$	$\mathbf{f(n)}$	$\lvert\pi - \mathbf{f(n)}\rvert$
0	3	$0.14 \leqslant 10^{-0}$
1	3.1	$0.04 \leqslant 10^{-1}$
2	3.14	$0.001 \leqslant 10^{-2}$
10	3.1415926535	$0.9 \cdot 10^{-10} \leqslant 10^{-10}$

From discrete to real computability

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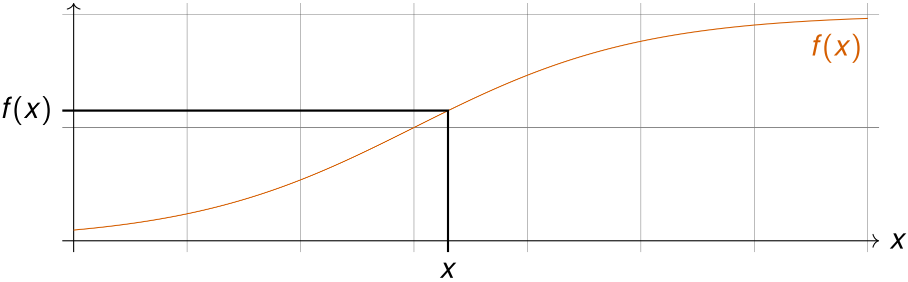
Beware : there exists uncomputable real numbers !

$$x = \sum_{n \in \Gamma} 2^{-n}, \qquad \Gamma = \{n : \text{the } n^{th} \text{ Turing machine halts}\}$$

From discrete to real computability



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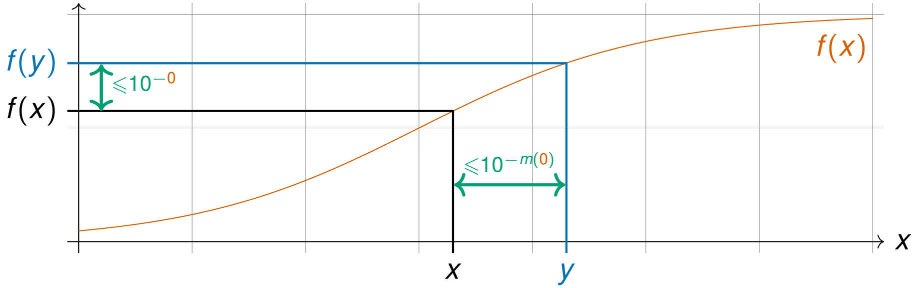


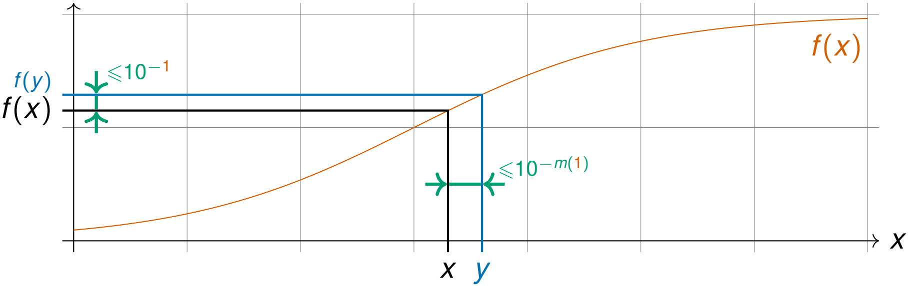
Definition (Computable function)

$f : [a, b] \to \mathbb{R}$ is computable iff $\exists\ m : \mathbb{N} \to \mathbb{N}$, computable functions such that :

$$|x - y| \leqslant 10^{-m(n)} \Rightarrow |f(x) - f(y)| \leqslant 10^{-n} \qquad x, y \in \mathbb{R}, n \in \mathbb{N}$$

m : modulus of continuity

From discrete to real computability



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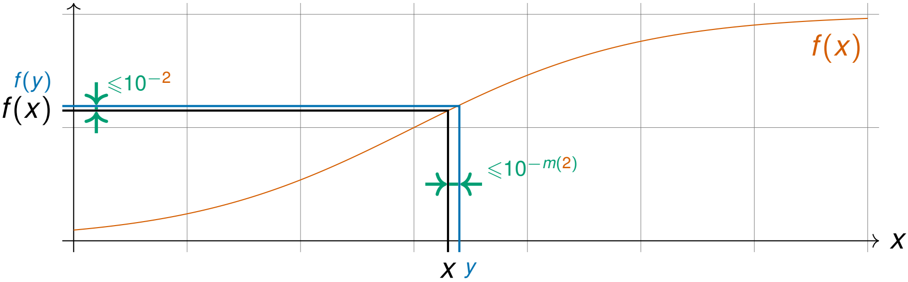

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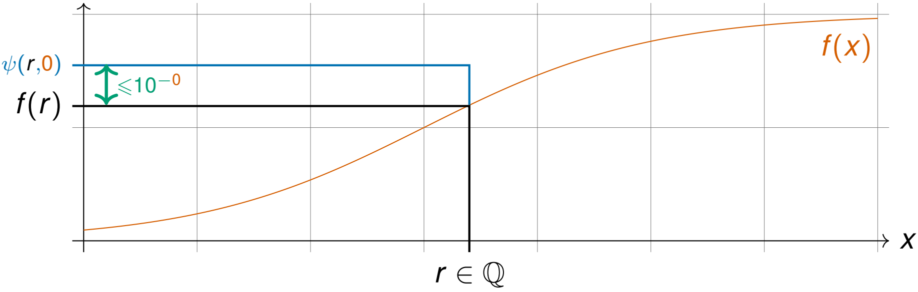

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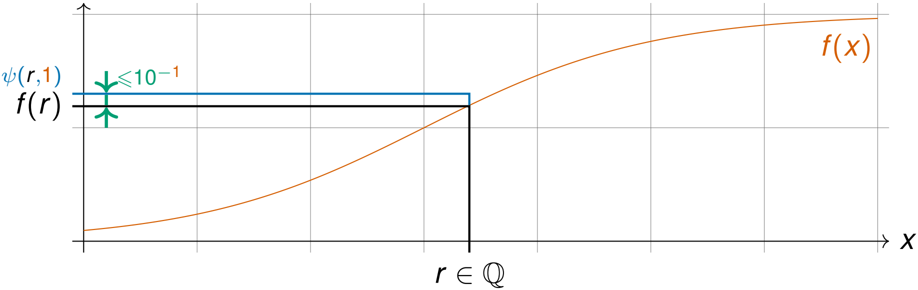

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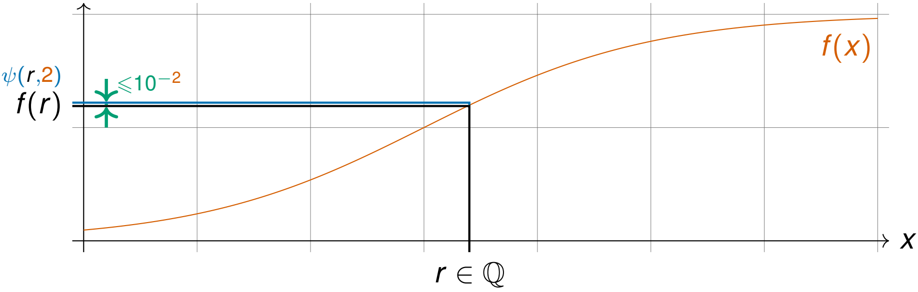

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Polytime complexity

Add “polynomial time computable” everywhere.

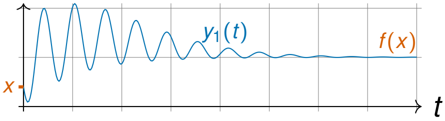
Equivalence with computable analysis

Definition (Bournez et al, 2007)

f **computable by GPAC** if $\exists p$ polynomial such that $\forall x \in [a,b]$

$$y(0) = (x, 0, \ldots, 0) \qquad y'(t) = p(y(t))$$

satisfies $|f(x) - y_1(t)| \leqslant y_2(t)$ et $y_2(t) \xrightarrow[t\to\infty]{} 0$.



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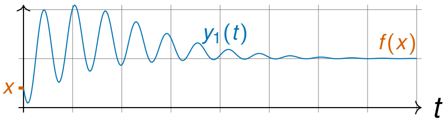

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Theorem (Bournez et al, 2007)

$f : [a, b] \to \mathbb{R}$ *computable (Computable Analysis)* $\Leftrightarrow$ *f computable by GPAC*

Complexity of analog systems

- Turing machines : $T(x)$ = number of steps to compute on x

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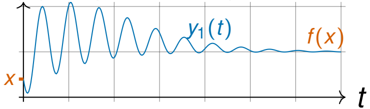
- GPAC :

Tentative definition

$T(x) = ??$

$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$



Complexity of analog systems

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Tentative definition

$T(x, \mu) =$

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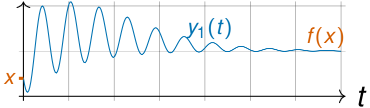

Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x
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Tentative definition

$T(x, \mu) =$ first time t so that $|y_1(t) - f(x)| \leqslant e^{-\mu}$

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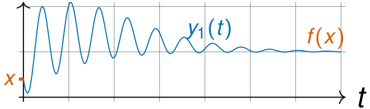

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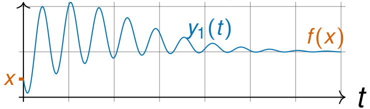
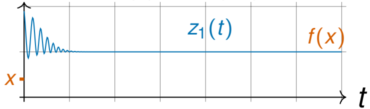


$\rightsquigarrow$

$z(t) = y(e^t)$



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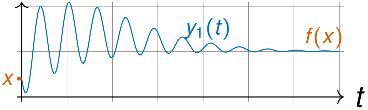

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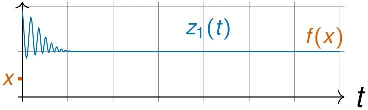

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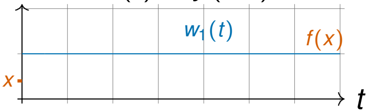

Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x
- GPAC : time contraction problem $\rightarrow$ **open problem**

Tentative definition

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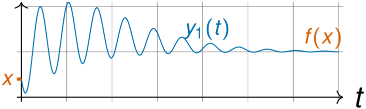

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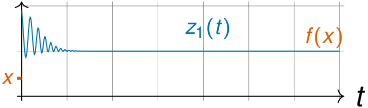

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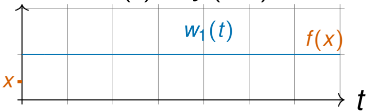
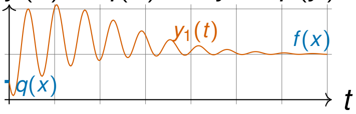


Something is wrong...

All functions have constant time complexity.

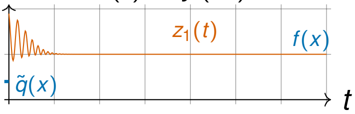
Time-space correlation of the GPAC

$y(0) = q(x) \qquad y' = p(y)$



$\rightsquigarrow$

$z(t) = y(e^t)$



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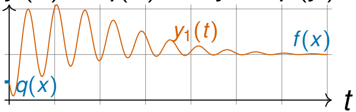

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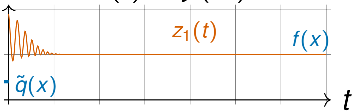
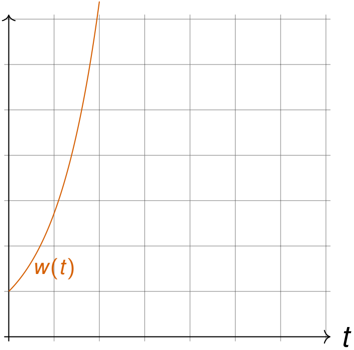


extra component : $w(t) = e^t$



Time-space correlation of the GPAC

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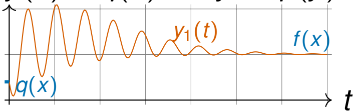

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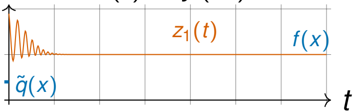

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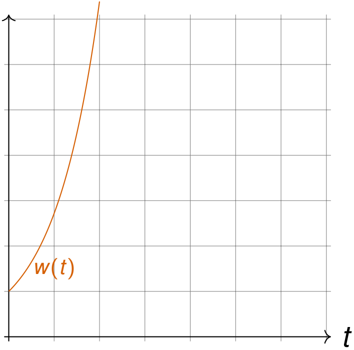


Observation

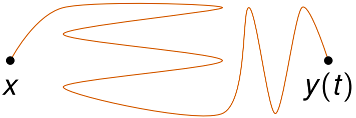
Time scaling costs "space".

$\rightsquigarrow$

Time complexity for the GPAC must involve time and space !

Complexity of solving polynomial ODEs

$$y(0) = x \qquad y'(t) = p(y(t))$$



Complexity of solving polynomial ODEs

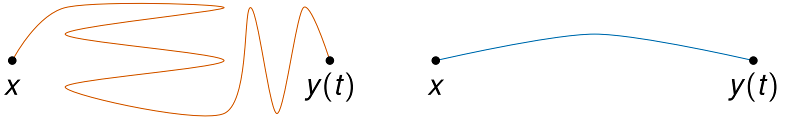
$$y(0) = x \qquad y'(t) = p(y(t))$$

Theorem

If $y(t)$ exists, one can compute p, q such that $\left|\frac{p}{q} - y(t)\right| \leqslant 2^{-n}$ in time

$$\operatorname{poly}(\textit{size of } x \textit{ and } p, n, \ell(t))$$

where $\ell(t) \approx$ length of the curve (between x and $y(t)$)

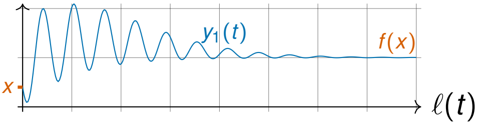


length of the curve = complexity = ressource

Characterization of real polynomial time

Definition : $f : [a, b] \to \mathbb{R}$ in ANALOG-P$_{\mathbb{R}}$ $\Leftrightarrow \exists p$ polynomial, $\forall x \in [a, b]$

$$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$$



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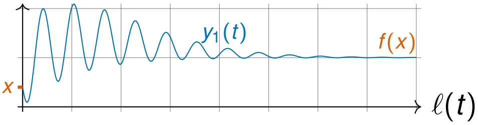
1. $|y_1(t) - f(x)| \leqslant 2^{-\ell(t)}$

   «greater length $\Rightarrow$ greater precision»

2. $\ell(t) \geqslant t$

   «length increases with time»



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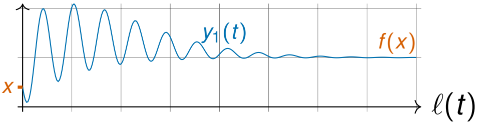


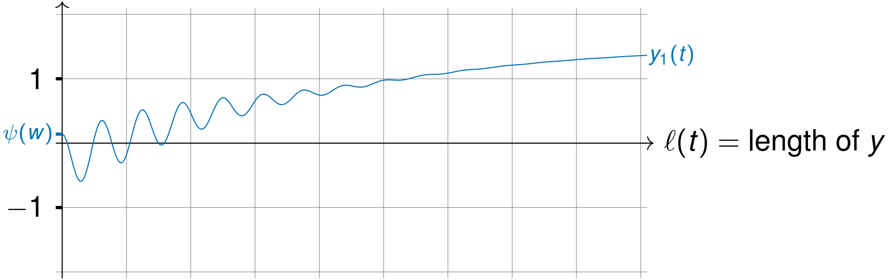
Theorem

$f : [a, b] \to \mathbb{R}$ computable in polynomial time $\Leftrightarrow$ $f \in$ ANALOG-$\mathrm{P}_{\mathbb{R}}$.

Characterization of polynomial time

Definition : $\mathcal{L} \in$ ANALOG-PTIME $\Leftrightarrow \exists p$ polynomial, $\forall$ word w

$$y(0) = (\psi(w), |w|, 0, \ldots, 0) \qquad y' = p(y) \qquad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}$$



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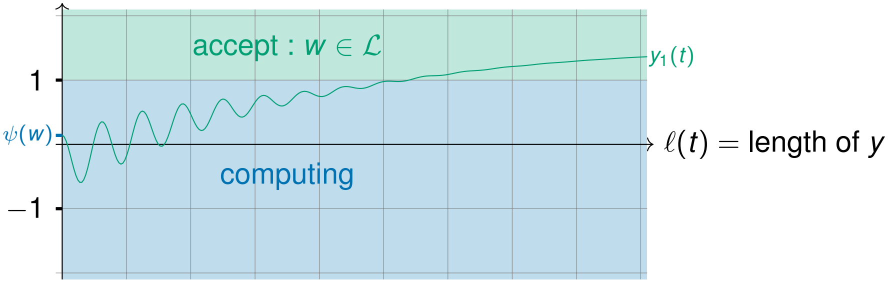

satisfies

1. if $y_1(t) \geqslant 1$ then $w \in \mathcal{L}$

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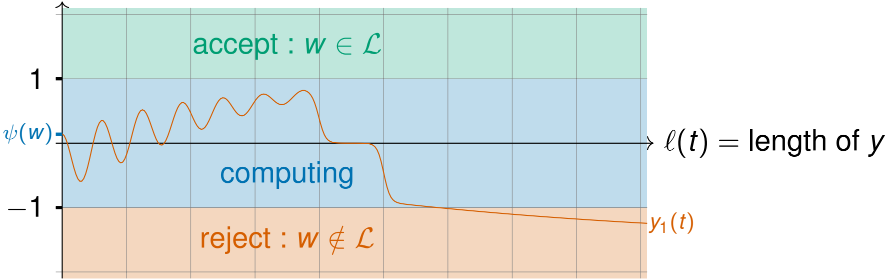

satisfies

2. if $y_1(t) \leqslant -1$ then $w \notin \mathcal{L}$

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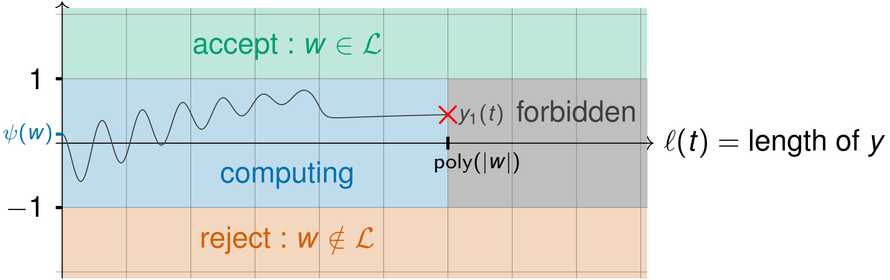

satisfies

3. if $\ell(t) \geqslant \text{poly}(|w|)$ then $|y_1(t)| \geqslant 1$

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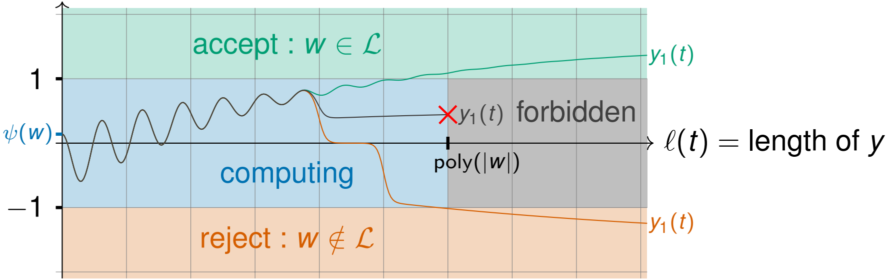
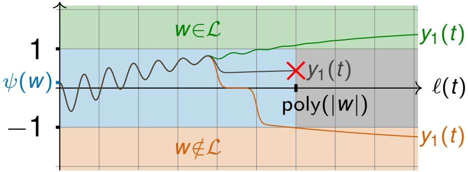


Theorem

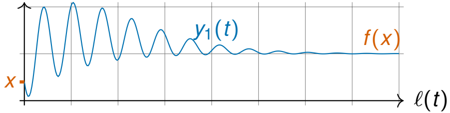
PTIME = ANALOG-PTIME

Summary

ANALOG-PTIME



ANALOG-P$_\mathbb{R}$



Theorem

- $\mathcal{L} \in$ PTIME *of and only if* $\mathcal{L} \in$ ANALOG-PTIME
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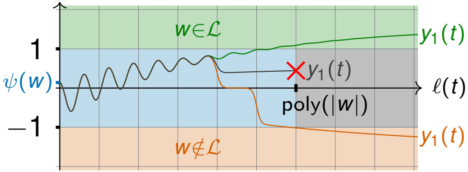
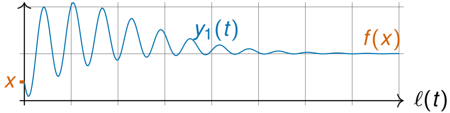
- Analog complexity theory based on length
- Time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME

Summary

ANALOG-PTIME



ANALOG-$P_{\mathbb{R}}$



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- Time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME
- Only rational coefficients needed

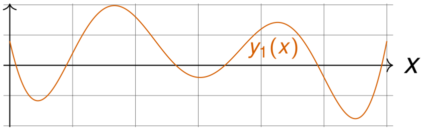
In the remaining time...

An applications of the “technology” we have developed :

Universal differential equation

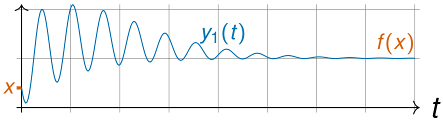
Universal differential equations

Generable functions



"Real-time" computability :
subclass of analytic functions

Computable functions



"Asymptotic" computability :
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Universal differential equations

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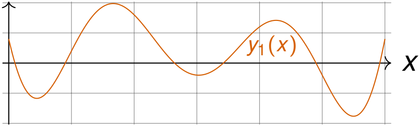

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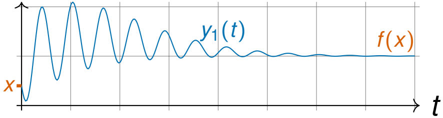

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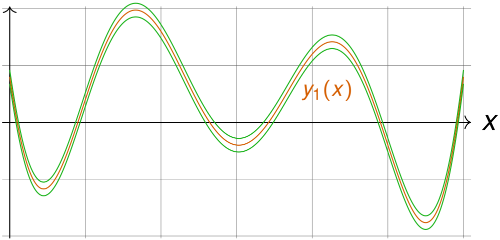
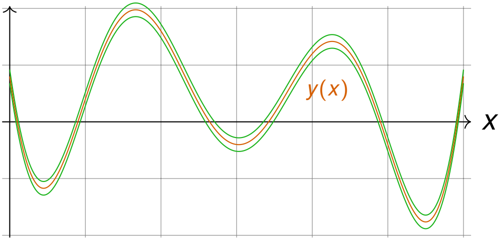


“Real-time” approximability : ? ?

Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

For any continuous functions f and ε, there exists $y : \mathbb{R} \to \mathbb{R}$ solution to

$$\begin{aligned} 3y'^4 y'' y''''^2 \quad & -4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ & -12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 \qquad = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

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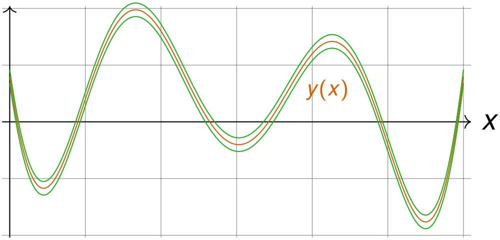

Theorem (Rubel, 1981)

There exists a ***fixed*** *polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to*

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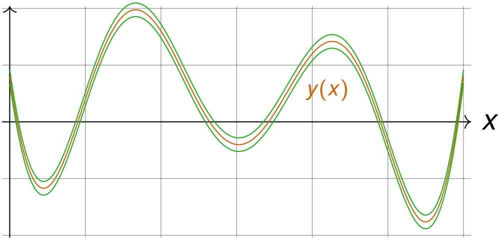

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Problem : this is «weak» result.

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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In fact, this is fundamental for Rubel's proof to work !

- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

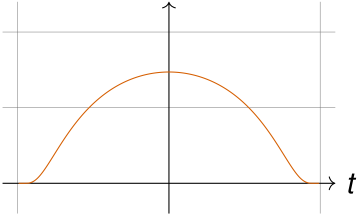
Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : unique solution

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

$$\text{It satisfies} \quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$$



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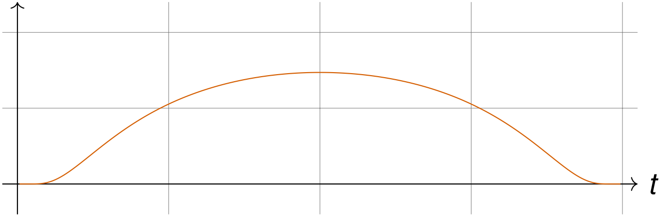
Translation and rescaling :



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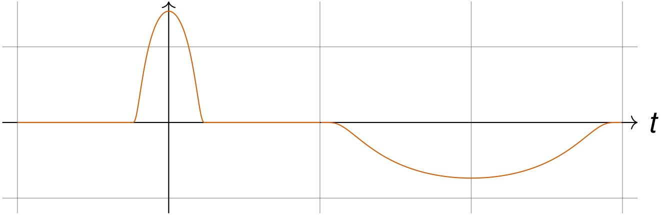
- Can glue together arbitrary many such pieces



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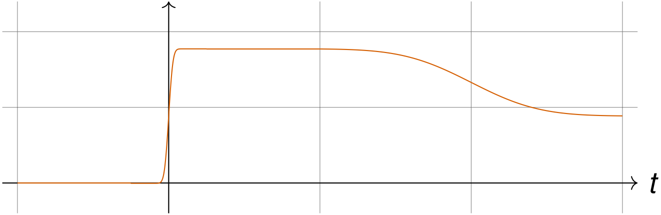
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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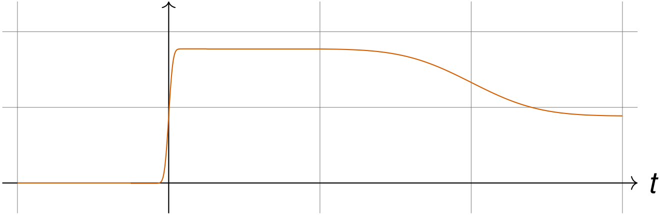
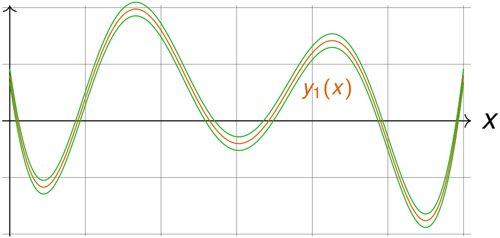


Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

Universal initial value problem (IVP)



Theorem (A truly universal ODE)

There exists a ***fixed*** *(vector of) polynomial p such that for any continuous functions f and* ε*, there exists* $\alpha \in \mathbb{R}^d$ *such that*

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a ***unique solution*** $y : \mathbb{R} \to \mathbb{R}^d$ *and* $\forall t \in \mathbb{R}$*,*

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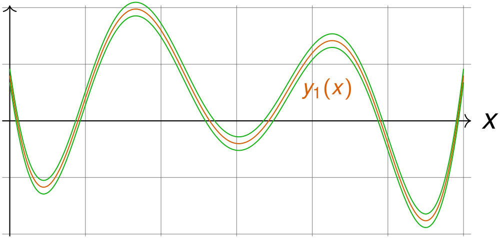

Notes :

- **system** of ODEs,
- y is analytic,
- we need $d \approx 300$.

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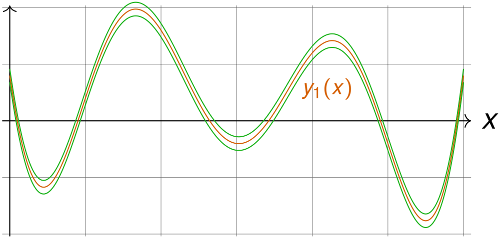

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*has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,*

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Remark : α is usually transcendental, but computable from f and ε

Future work

Reaction networks :

- chemical
- enzymatic

$$y' = p(y)$$

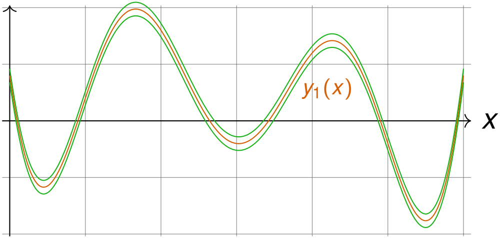
?

$$y' = p(y) + e(t)$$

- Finer time complexity (linear)
- Nondeterminism
- Robustness
- « Space» complexity
- Other models
- Stochastic

Backup slides

Universal DAE revisited



Theorem

*There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that*

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

*has a **unique analytic solution** and this solution satisfies such that*

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Chemical Reaction Networks

Definition : a **reaction system** is a finite set of

- molecular species $y_1, \ldots, y_n$
- reactions of the form $\sum_i a_i y_i \xrightarrow{f} \sum_i b_i y_i \qquad (a_i, b_i \in \mathbb{N},\ f = \text{rate})$

Example :

$$\begin{array}{rcccl} 2H_2 & + & O & \to & 2H_2O \\ C & + & O_2 & \to & CO_2 \end{array}$$

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Theorem (Folklore)

Every polynomial ODE can be rewritten as a quadratic ODE.

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Theorem (CMSB, joint work with François Fages, Guillaume Le Guludec)

Elementary mass-action-law reaction system on finite universes of molecules are Turing-complete under the differential semantics.

Notes :

- proof preserves polynomial length
- in fact the following elementary reactions suffice :

$$\varnothing \xrightarrow{k} x \qquad x \xrightarrow{k} x + z \qquad x + y \xrightarrow{k} x + y + z \qquad x + y \xrightarrow{k} \varnothing$$