Provable Dual Attacks on Learning with Errors

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Learning with Error (LWE)

Fundamental problem for lattice-based cryptography

- n: dimension of secret
- m: number of samples
- χ_e: error distribution over $\mathbb{Z}_q$
- q: prime number
- $\mathbf{s} \in \mathbb{Z}_q^n$: secret

$\mathrm{LWE}(m, \mathbf{s}, \chi_e)$ distribution

Sample $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ *uniformly at random* and $\mathbf{e} \in \mathbb{Z}_q^m$ *according to* χ_e^m.
Output $(\mathbf{A}, \mathbf{b})$ where $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$.

Search LWE problem

Given $(\mathbf{A}, \mathbf{b})$ sampled from $\mathrm{LWE}(m, \mathbf{s}, \chi_e)$, recover (part of) $\mathbf{s}$.

In this paper:

- no assumption on $\mathbf{s}$ and χ_e
- $m \approx 2n$ (more on that at the end)

Dual attacks: brief history and controversy

Two main types of attacks: primal and dual.

[GJ21] dual attack with sieving, DFT, suggested modulus switching

[MAT22] formal analysis of dual attack with sieving + modulus switching

- $\rightsquigarrow$ claims comparable with best primal attacks (in some regime)
- $\rightsquigarrow$ correctness relies on statistical assumptions: do these really hold?

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- Shows that it does not hold for [MAT22]'s parameters
- Concludes that [MAT22]'s result is unsubstantiated

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Open question: is [DP23a]'s simplified assumption really equivalent to [MAT22]'s key assumption? $\rightsquigarrow$ more on this later

Contributions

Main result

Completely formal, non-asymptotic analysis of a simplified dual attack.

- no assumptions $\leadsto$ no controversy
- makes it clear in which parameter regime the attack works
  $\leadsto$ almost complementary with [DP23a]'s contradictory regime in our simplified setting
- uses discrete Gaussian sampling (DGS) instead of sieving

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Other contributions:

- Quantum version of the algorithm with non-trivial speed up based on ideas from [AS22]
- Improved analysis of DGS with BKZ reduced basis based on the Monte Carlo Markov Chain sampler [WL19]
- Complexity estimates for concrete parameters (Kyber)

Our dual attack on LWE: high-level

Given $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$, split secret into two parts ($n = n_{\text{guess}} + n_{\text{dual}}$):

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Consider the lattice

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Good guess: $\tilde{\mathbf{s}}_{\text{guess}} = \mathbf{s}_{\text{guess}}$

$f(\mathbf{b} - \mathbf{A}_{\text{guess}}\tilde{\mathbf{s}}_{\text{guess}}) \approx g(\operatorname{dist}(\mathbf{e}, L)) = g(\|\mathbf{e}\|)$ if $\mathbf{e}$ is sufficiently small.

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Bad guess: $\tilde{\mathbf{s}}_{\text{guess}} \neq \mathbf{s}_{\text{guess}}$

For most $\mathbf{A}$, $\operatorname{dist}(\mathbf{A}_{\text{guess}}(\mathbf{s}_{\text{guess}} - \tilde{\mathbf{s}}_{\text{guess}}) + \mathbf{e}, L) > \|\mathbf{e}\|$ if $\mathbf{e}$ is sufficiently small. So $f(\mathbf{b} - \mathbf{A}_{\text{guess}}\tilde{\mathbf{s}}_{\text{guess}}) \lessapprox g(\|\mathbf{e}\|)$.

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Summary: If $\mathbf{e}$ is sufficiently small and for most $\mathbf{A}$,

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  $\rightsquigarrow$ complicated to analyze, major source of problems in [MAT22] and leads to statistical assumptions
- BKZ + Gaussian sampler:
  $\rightsquigarrow$ well understood, $f(\mathbf{t}) \approx \rho_s(\mathrm{dist}(\mathbf{t}, L))$ [AR05]
  $\rightsquigarrow$ considered inefficient for dual attacks, maybe wrongly so!

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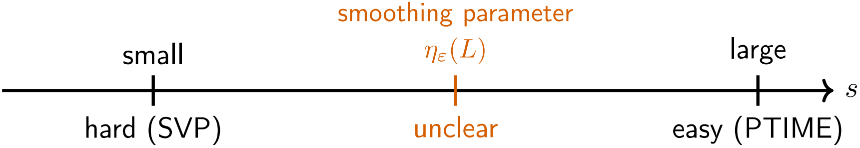

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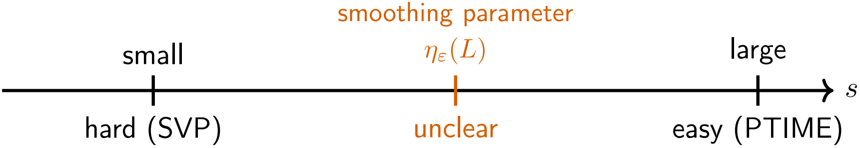

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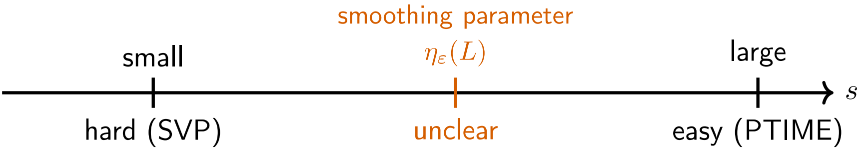

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1. Klein sampler: PTIME, s depends on basis but $s \geqslant \eta_\varepsilon(L)$ by construction $\rightsquigarrow$ not good enough
2. Monte Carlo Markov Chain (MCMC) sampler [WL19]: complexity and s depend on basis, no constraint on s
   - regime where $s < \eta_\varepsilon(L)$ and the sampler runs in exponential time
   - the generic complexity bound in [WL19] is not good enough
   - we improved it specifically for BKZ-reduced basis under GSA

Main result and working/contradictory regime

Main result (very informal)

Our dual attack works for most $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e})$ as long as $\|\mathbf{e}\| \leqslant \frac{1}{2}\lambda_1(L_q(\mathbf{A}))$.

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In [DP23a], the authors introduced a "contradictory regime" where dual attacks provably do not work. In our setting (simplified attack), this regime is roughly

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Take away (for simplified attack):

- [DP23a] + our work covers most of the parameter range
- Open question: what happens for $\frac{1}{2} \leqslant \frac{\|\mathbf{e}\|}{\lambda_1(L_q(\mathbf{A}))} \leqslant 1$?

Complexity estimates

Our attack does not have modulus switching $\rightsquigarrow$ not competitive

Scheme	attack	m	n_{guess}	n_{dual}	β
Kyber512	185	1013	15	497	550
Kyber768	273	1469	23	745	870
Kyber1024	376	2025	31	993	1230

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We estimated the complexity of a hypothetical extension of our attack with modulus switching (MS):

Scheme	Our attack	MS	MATZOV
Kyber512	185	141	143
Kyber768	273	202	200
Kyber1024	376	279	264

- promising but unproven, most likely too optimistic
- validates the approach of BKZ + MCMC DGS sampling

Targets and related works

Given $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$,

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where $f = f_{\mathcal{X}}$ for some (sampled) dual vectors $\mathcal{X} \subseteq \widehat{L}$.

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$$\Pr_{\mathcal{X}, \mathbf{t}^{(\mathrm{i})} \sim \mathcal{U}(\mathbb{Z}^m/L)}\left[f(\mathbf{e}) > f(\mathbf{t}^{(\mathrm{i})}), i = 1, \ldots, q^{n_{\text{guess}}}\right]. \tag{2}$$

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Later [CDMT24] and [DP23b] analyzed the distribution of $f(\mathbf{t})$ when $\mathbf{t} \sim \mathcal{U}(\mathbb{Z}^m/L)$ and $\mathcal{X}$ comes from sieving in $\widehat{L}$.

Open question: (1) is NOT equivalent to (2), how do they compare?

Conclusion and future work

- strong foundation for provable dual attacks with no assumptions
- BKZ + MCMC DGS sampling seems competitive with BKZ + sieving but simpler to analyze
- promising complexity estimates
- quantum algorithm with non-trivial speed up

Open questions:

- analyze modulus switching or coding theory-based dimension reduction from [CST22]
- close the gap between working and contradictory regime
- make the attack work with $m = n$ samples by using

$$\left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^m \times \mathbb{Z}^{n_{\text{dual}}} : \mathbf{A}_{\text{dual}}^T \mathbf{x} = \mathbf{y} \bmod q\right\}$$

instead of dual lattice

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