Solving the Shortest Vector Problem in $2^{0.63269n+o(n)}$ time on Random Lattices

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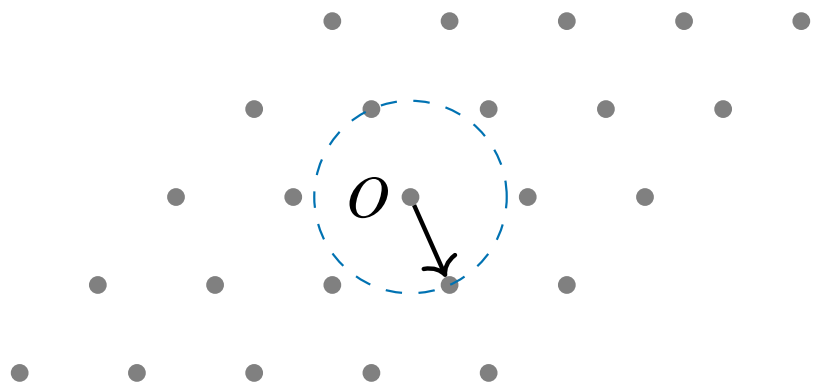

Shortest Vector Problems

Definition (Shortest Vector Problem, SVP)

On lattice $\mathcal{L}$ output a vector $\mathbf{y} \in \mathcal{L} \setminus \{0\}$ with $\|\mathbf{y}\| \leqslant \lambda_1(\mathcal{L})$.

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On lattice $\mathcal{L}$ output a vector $\mathbf{y} \in \mathcal{L} \setminus \{0\}$ with $\|\mathbf{y}\| \leqslant \gamma \cdot \lambda_1(\mathcal{L})$.

- fundamental problem
- building block of lattice reduction algorithms (e.g. BKZ)

Significant gap between provable and heuristic for SVP (2^n vs $2^{0.292n}$)

- provable algorithm work on **all lattices** (worst case)
- heuristic algorithms work for **most lattices** (average case)

Motivation

Bridge the gap between heuristic and provable results: provable resultats for random lattices.

Random Lattices in Cryptography

Lattices in cryptography are random:

- random q-ary:
  - ring: integers, polynomials, modules
  - distribution: uniform, Gaussian
- random real lattices:
  - formally defined using a Haar measure
  - in some sense, the limit of sampling uniformly an integer lattice of fixed volume and rescaling

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Main contribution

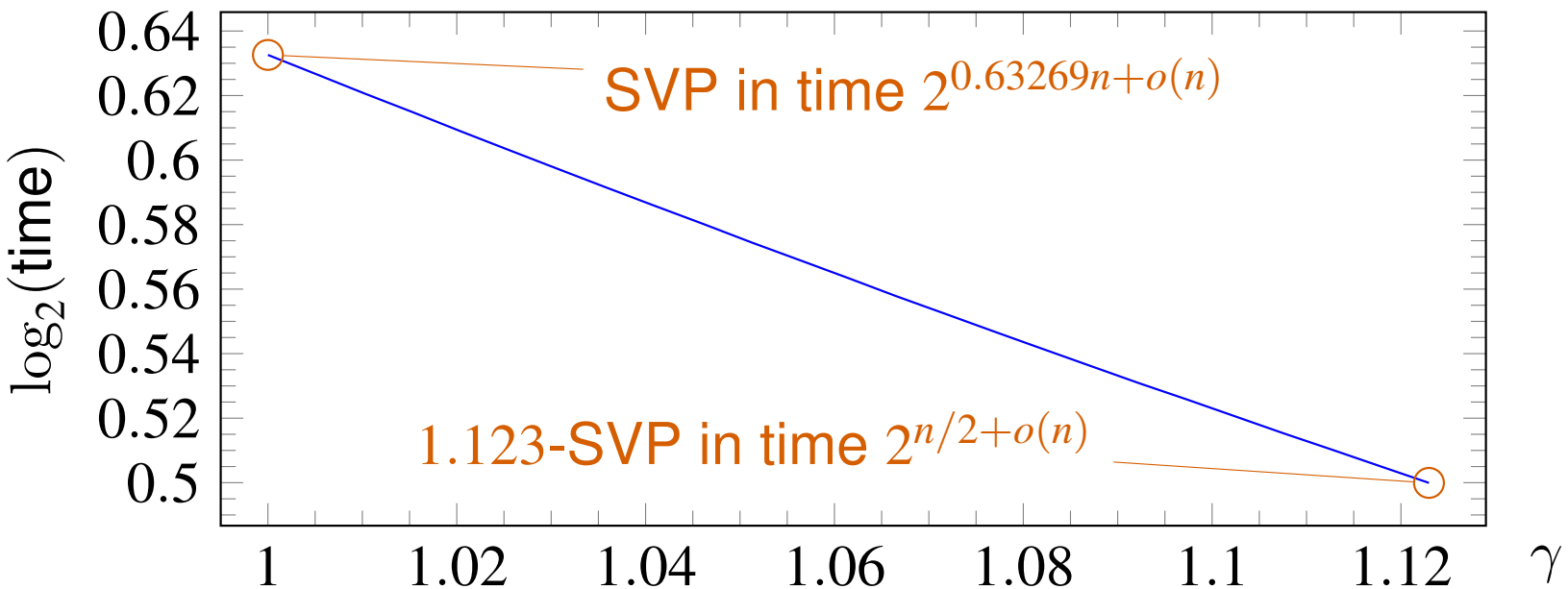
Solve SVP faster on real random lattices using discrete Gaussian sampling.

Why real lattices? $\rightsquigarrow$ Easiest to work with technically.

Contributions

SVP on random lattices

Algorithm that, for every $n \geqslant 1$ and $\gamma \in [1, 1.1230]$, solves γ-SVP on almost all* lattices in time $e^{o(n)}(\frac{\gamma^2}{2}e^{-\gamma^2/2e})^{-n/2}$ and space $2^{n/2+o(n)}$.



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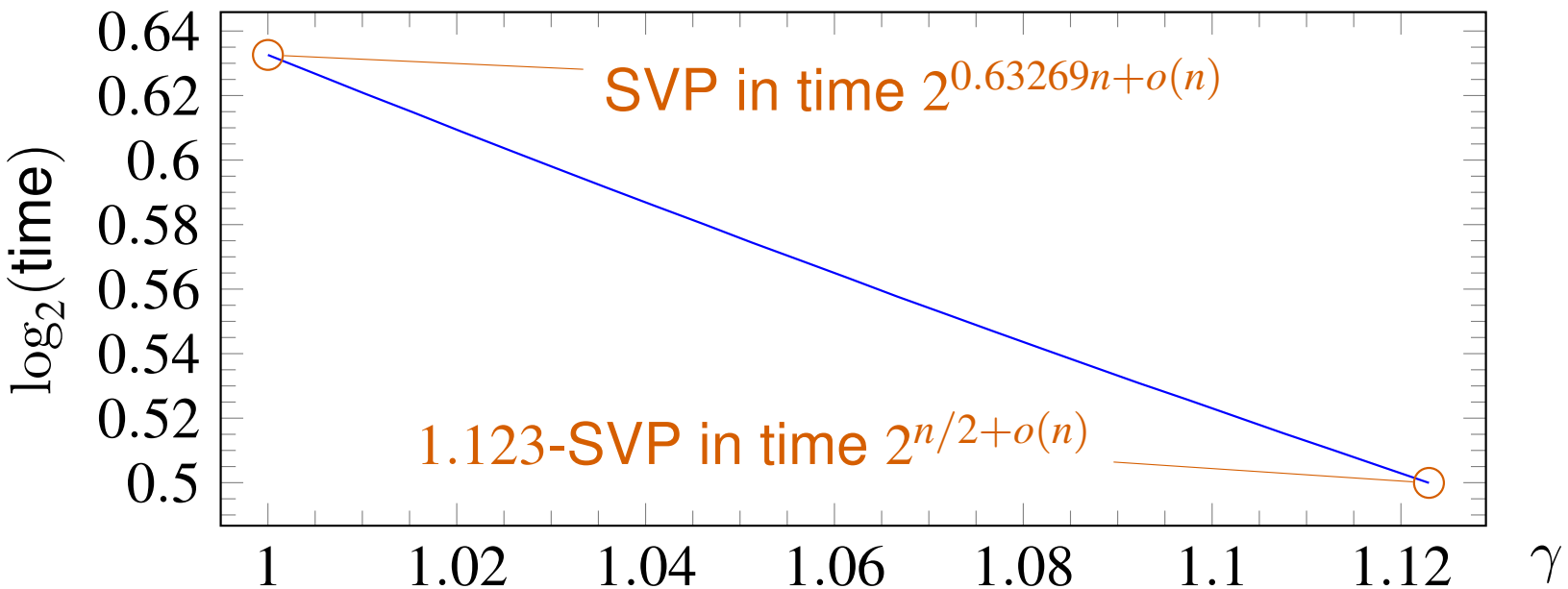
More generally, trade-off between complexity and fraction of lattices.



*Fraction $1 - 1/\text{poly}(n)$ of real lattices, can do any $1 - \varepsilon$, complexity depends on ε.

Comparison with state of the art

Problem	Time	Reference	Type
SVP	$2^{0.292n+o(n)}$	[BDGL16]	Heuristic (sieving)
SVP	$2^{n+o(n)}$	[ADRS15]	Provable
SVP	$2^{0.63269n+o(n)}$	This work	Provable (random)
$O(1)$-SVP	$2^{0.802n+o(n)}$	[WLW15]	Provable
100-SVP	$2^{0.824n+o(n)}$	[AUV19]†	Provable
1.123-SVP	$2^{n/2+o(n)}$	This work	Provable (random)
$\tilde{O}(\sqrt{n})$-SVP	$2^{n/2+o(n)}$	[ALS21]	Provable

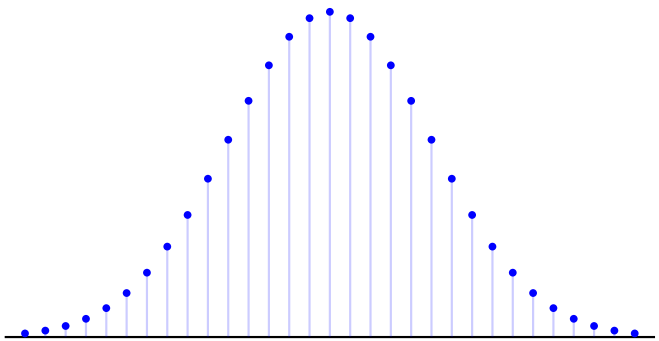
†Constant in [WLW15] analysed by [AUV19], 100-SVP is just one example.

Discrete Gaussian Sampling

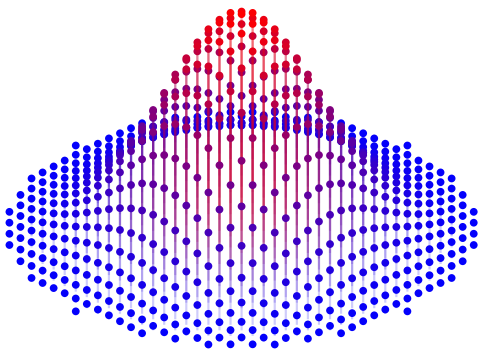
$$\rho_s(\boldsymbol{x}) = \exp\left(-\pi \frac{\|\boldsymbol{x}\|^2}{s^2}\right), \qquad D_{\mathcal{L},s}(\boldsymbol{x}) = \frac{\rho_s(\boldsymbol{x})}{\rho_s(\mathcal{L})}, \qquad \boldsymbol{x} \in \mathbb{R}^n, s > 0.$$

Discrete Gaussian Distribution

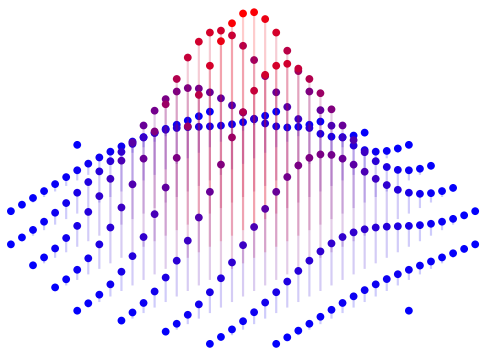
On lattice $\mathcal{L}$ with parameter s: probability of $\boldsymbol{x} \in \mathcal{L}$ is $D_{\mathcal{L},s}(\boldsymbol{x})$.



$\mathcal{L} = \mathbb{Z}, s = 7$



$\mathcal{L} = \mathbb{Z}^2, s = 7$



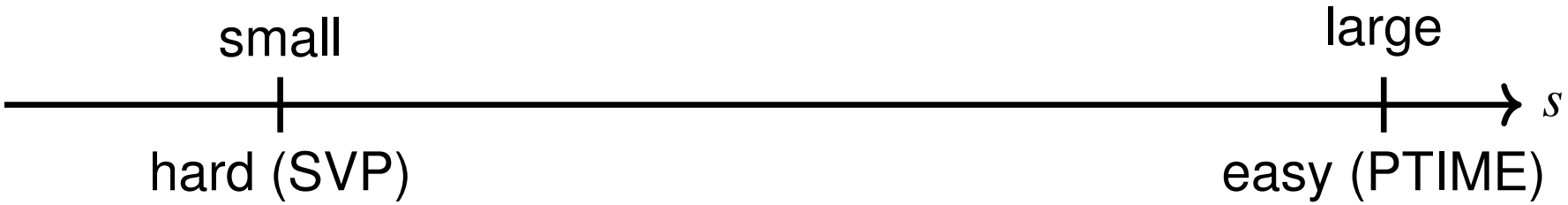
$\mathcal{L} = \mathbb{Z} \times 4\mathbb{Z}, s = 7$

Discrete Gaussian Sampling (DGS)

- **input:** $\mathcal{L}$ and s
- **output:** random $\boldsymbol{x} \in \mathcal{L}$ according to $D_{\mathcal{L},s}$.

Complexity of Discrete Gaussian Sampling

Sampling from the discrete Gaussian over $\mathcal{L}$ with parameter s:



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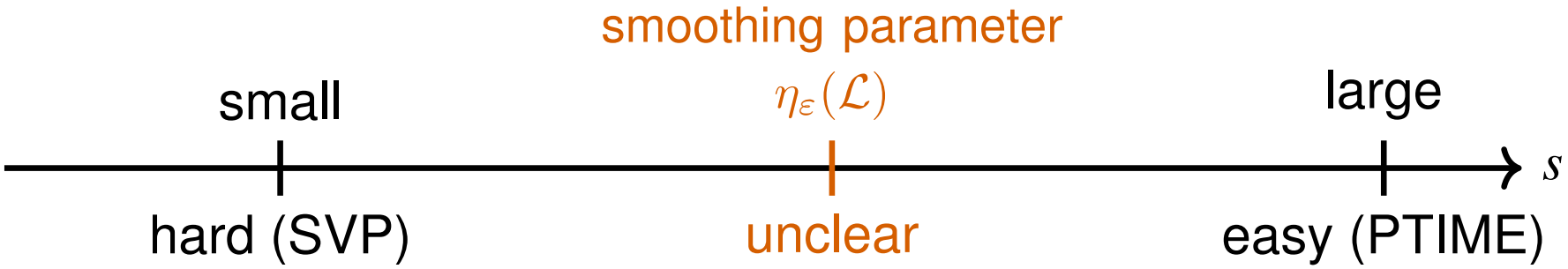

Smoothing parameter

For $\varepsilon > 0$, $\eta_\varepsilon(\mathcal{L}) = \inf\left\{s > 0 : \rho_{1/s}(\widehat{\mathcal{L}}) \leqslant 1 + \varepsilon\right\}$.

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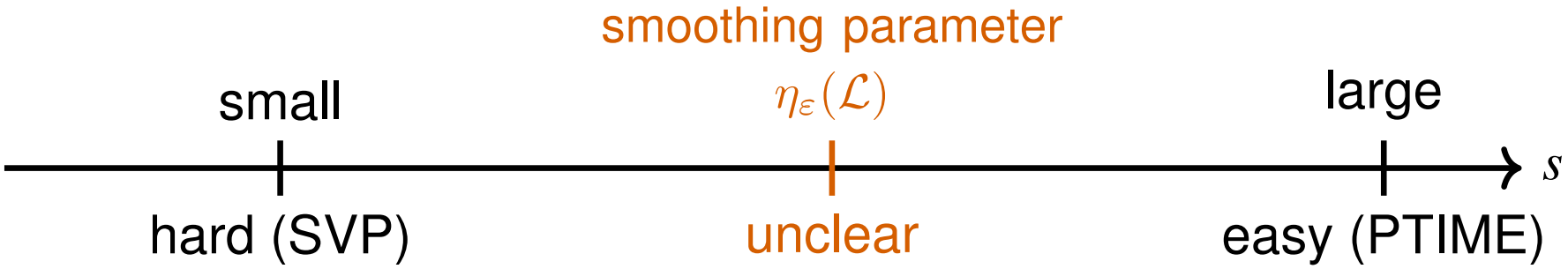

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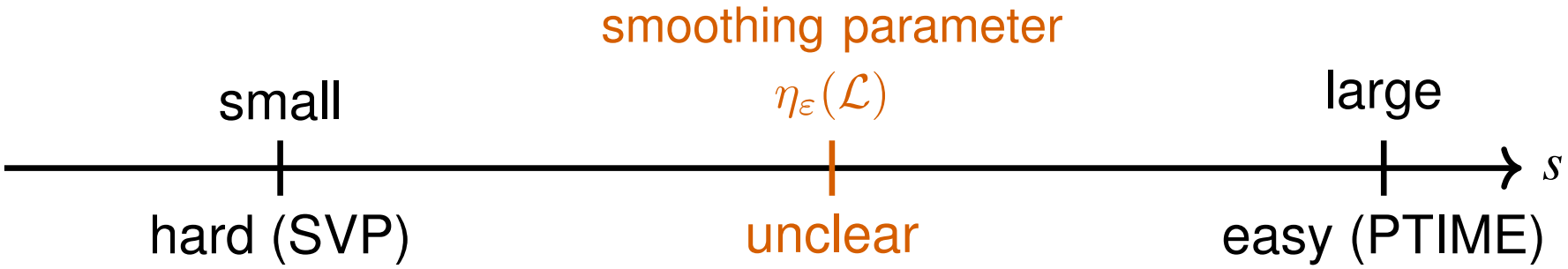

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Theorem ([ADRS15])

For any $\mathcal{L} \subset \mathbb{R}^n$ and $s \geq \sqrt{2}\eta_{1/2}(\mathcal{L})$, can get $2^{n/2}$ samples from a distribution $2^{-\Omega(n^2)}$-close to $D_{\mathcal{L},s}$ in expected time $2^{n/2+o(n)}$.

High-level view of the algorithm

Algorithm for γ-SVP

Sample N_γ vectors from $D_{\mathcal{L},\sqrt{2}\eta_{1/2}(\mathcal{L})}$, keep the shortest nonzero.

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Analysis requires:

- $\eta_{1/2}(\mathcal{L})$: smoothing parameter
- nonzero:

$$\Pr_{\mathbf{x}\sim D_{\mathcal{L},s}}[\mathbf{x}=0] = \frac{1}{\rho_s(\mathcal{L})}$$

- N_γ: estimate

$$\Pr_{\mathbf{x}\sim D_{\mathcal{L},s}}[\|\mathbf{x}\| \leqslant r] = \frac{\rho_s(\mathcal{L}\cap B_n(r))}{\rho_s(\mathcal{L})}, \qquad r = \gamma\cdot\lambda_1(\mathcal{L})$$

- $\lambda_1(\mathcal{L})$: length of SVP

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Non-tight estimates in **general**, but can do better for random lattices

Random Real Lattices

High-level intuition:

- Consider lattice basis modulo scale: $\mathrm{vol}(\mathcal{L}) = 1 \rightsquigarrow \mathrm{SL}_n(\mathbb{R})$
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Theorem ([Sie45])

There is a unique "natural" probability measure μ_n on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$.

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Averaging result ([Sie45])

Let $n \geqslant 1$ and f be a Lebesgue integrable function on $\mathbb{R}^n$, then

$$\mathbb{E}_{\mathcal{L}\sim\mu_n}\left[\sum_{\mathbf{x}\in\mathcal{L}\setminus\{0\}} f(\mathbf{x})\right] = \int_{\mathbb{R}^n} f(\mathbf{x})\mathrm{d}\,\lambda(\mathbf{x}).$$

where λ denotes the usual Lebesgue measure on $\mathbb{R}^n$

Averaging results: Gaussian mass

For any $s > 0$,

$$\mathbb{E}_{\mathcal{L}\sim\mu_n}[\rho_s(\mathcal{L})] = 1 + \int_{\mathbb{R}^n} \rho_s(\mathbf{x}) \mathrm{d}\,\lambda(\mathbf{x}) = 1 + s^n.$$

Can obtain a bound on $\rho_s(\mathcal{L})$ and $\eta_\varepsilon(\mathcal{L})$ using Markov's inequality.

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Similar results for other models:

Model	**Method**	**Ref**
q-ary, Gaussian entries	estimate λ_1, bound η_ε	[KNSW20]
q-ary, uniform entries	Siegel-like* method	[LLBS14]
q-ary / cyclotomic field	ad-hoc calculation estimate λ_1, bound η_ε	[CPSWX20; LPR13] [SS11]

*A generic Siegel-like averaging result for q-ary was already shown in [Loe97].

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However, Markov inequality is not very tight...

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Averaging results: variance

The averaging result of Siegel is not always sufficient. A higher-order version was shown* by Mac Beath and Rogers [MR58]. However the statement is complicated.

Corollary ([MR58])

Let $n \geqslant 2$ and f be Lebesgue integrable on $\mathbb{R}^n$, then

$$\mathbb{V}_{\mathcal{L}\sim\mu_n}\left[\sum_{\mathbf{x}\in\mathcal{L}} f(\mathbf{x})\right] = \frac{1}{\zeta(n)} \sum_{\alpha\in\mathbb{N}\setminus\{0\}} \sum_{\beta\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}^n} f(\alpha\mathbf{x}) f(\beta\mathbf{x}) \mathrm{d}\,\lambda(\mathbf{x}).$$

*There is an error in the original proof of Rogers, noticed and fixed in [Kim24].

Gaussian mass and smoothing parameter

Theorem (Gaussian mass, simplified)

For any $s > 0$ and $\alpha > 0$

$$\Pr_{\mathcal{L}\sim\mu_n}[|\rho_s(\mathcal{L}) - 1 - s^n| > \alpha] \leqslant \frac{2^{-n/2}s^n}{\alpha^2}.$$

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Corollary (Smoothing parameter, simplified)

For any $\varepsilon > 0$, let $s_\varepsilon = \left(\frac{\varepsilon+1+\sqrt{2\,\varepsilon+1}}{\varepsilon^2}\right)^{1/n}$. Then

$$\Pr_{\mathcal{L}\sim\mu_n}[\eta_\varepsilon(\mathcal{L}) > s_\varepsilon] \leqslant 2^{-n/2}.$$

Open question: smoothing parameter

For any $\varepsilon > 0$, almost all lattices $\mathcal{L}$ satisfy that

$$\eta_\varepsilon(\mathcal{L}) \leqslant \left(\frac{\varepsilon + 1 + \sqrt{2\,\varepsilon + 1}}{\varepsilon^2}\right)^{1/n} \sim_{\varepsilon \to 0} 2^{1/n} \varepsilon^{-2/n}$$

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Combine with the bound on $\lambda_1(\mathcal{L})$:

$$\lambda_1(\mathcal{L}) \eta_\varepsilon(\widehat{\mathcal{L}}) \leqslant \sqrt{\tfrac{n}{2\pi e}} \cdot \varepsilon^{-2/n} \tag{1}$$

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Compare with the unconditional result of [ADRS15]:

$$\lambda_1(\mathcal{L}) \eta_\varepsilon(\widehat{\mathcal{L}}) < \sqrt{\frac{\beta(\mathcal{L})^2 n}{2\pi e}} \cdot \varepsilon^{-1/n} \tag{2}$$

where $\beta(\mathcal{L}) \leqslant 2^{0.401}$ is the generalized kissing number [ACKS21].

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Discrepancy:

- $\varepsilon^{-2/n}$ in our case,
- $\varepsilon^{-1/n}$ in [ACKS21].

Good:

- suggest $\beta(\mathcal{L}) \approx 1$ for random $\mathcal{L}$

Conclusion and future work

Study of real random lattices:

- probabilistic bounds on $\rho_s(\mathcal{L})$, $\eta_\varepsilon(\mathcal{L})$ and $\rho_s(\mathcal{L} \cap B_n(r))$
- time/approx tradeoff for γ-SVP (constant γ regime)
- SVP and $\sqrt{\frac{n}{2\pi e}}$-HSVP in time $2^{0.63269n+o(n)}$
- 1.123-SVP in time $2^{n/2+o(n)}$

Open question: improve bound on η_ε to resolve discrepancy

- use Rogers averaging for higher order, difficult computations
- if we can solve this, can improve BDD results from [ACKS21]

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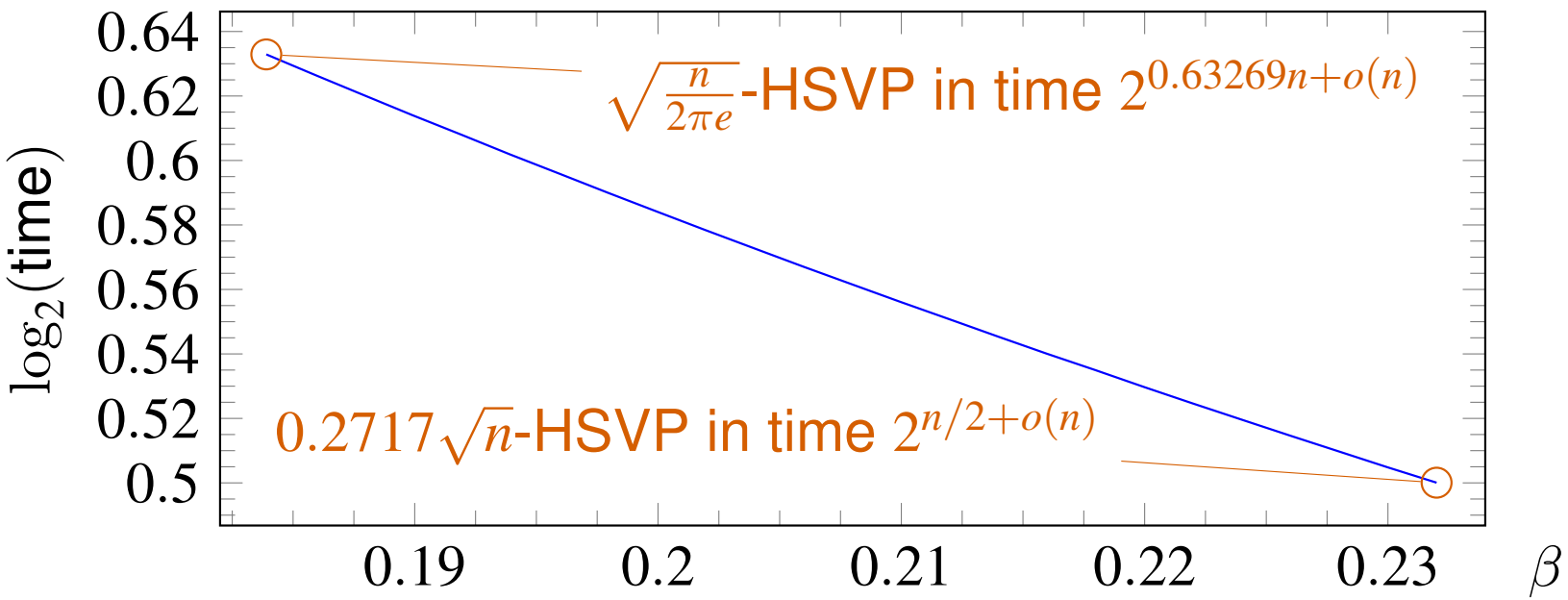
[WLW15] Wei Wei, Mingjie Liu, and Xiaoyun Wang. *Finding Shortest Lattice Vectors in the Presence of Gaps*.

Contributions (cont.)

HSVP on random lattices

Algorithm that, for every $n \geqslant 1$ and $\beta \in [\frac{1}{2e}, 0.2320]$, solves $\sqrt{\frac{n\beta}{\pi}}$-HSVP on almost all lattices in time $e^{o(n)}(\beta e^{1-\beta})^{-n/2}$ and space $2^{n/2+o(n)}$.



GapSVP

Promise version of SVP:

Definition (γ-GapSVP)

On lattice $\mathcal{L}$ and $r > 0$, accept if $\lambda_1(\mathcal{L}) \leqslant r$ and reject if $\lambda_1(\mathcal{L}) \geqslant \gamma r$.

Best provable result: $2^{n/2+o(n)}$-time algorithm for 1.93-GapSVP [ADRS15].

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Random real lattices: concentration bounds on $\lambda_1(\mathcal{L})$.

Corollary (Informal, was already folklore)

There is an algorithm that, for every $\gamma > 1$, $n \geqslant 1$ and on most lattices $\mathcal{L} \subset \mathbb{R}^n$, solves γ-GapSVP in polynomial time.

Algorithm: on input $\mathcal{L} \subseteq \mathbb{R}^n$ and $r > 0$, accept if $\mathrm{vol}(B_n)^{-1/n} < r\sqrt{\gamma}$ and reject otherwise.

Estimating the Smoothing Parameter

- Generic bound [ADRS15]:

$$\sqrt{\frac{\log(1/\varepsilon)}{\pi}} < \lambda_1(\mathcal{L})\eta_\varepsilon(\widehat{\mathcal{L}}) < \sqrt{\frac{\beta(\mathcal{L})^2 n}{2\pi e}} \cdot \varepsilon^{-1/n} \cdot (1 + o(1))$$

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  - $\beta(\mathcal{L}) \leqslant 2^{0.401}$ pessimistic (see later)

Definition (Smoothing Parameter Problem γ-GapSPP$_{\varepsilon_Y,\varepsilon_N}$)

On lattice $\mathcal{L}$: accept if $\eta_{\varepsilon_Y}(\mathcal{L}) \leqslant 1$, reject if $\eta_{\varepsilon_N}(\mathcal{L}) > \gamma$.

- Hard problem [CDLP13]:
  - $(1 + o(1))$-GapSPP$_{\varepsilon,\varepsilon} \in$ DTIME$(2^{O(n)}\text{polylog}(1/\varepsilon))$
  - reduction* from α-BDD to $\sqrt{n}/\alpha$-GapSPP$_{\text{negl}(n),1/\text{poly}(n)}$
  - quantum reduction* from LWE$_\alpha$ to $2\sqrt{n}/\alpha$-GapSPP$_{\text{negl}(n),1/\text{poly}(n)}$

*Result highly simplified for presentation.

Averaging results: higher order

The averaging result of Siegel is not always sufficient.

Generalized averaging result* ([MR58])

Let $1 \leqslant \ell \leqslant n-1$ and f be Lebesgue integrable on $\mathbb{R}^{n\times\ell}$, then

$$\int_{X_n} \sum_{\mathbf{M}\in \mathrm{LI}_{n,\ell}} f(\mathbf{AM})\mathrm{d}\,\mu(\mathbf{A})$$

$$= \zeta(n)\cdots\zeta(n-\ell+1)\int_{X_n}\sum_{\mathbf{P}\in\mathrm{PR}_{n,\ell}} f(\mathbf{AP})\mathrm{d}\,\mu(\mathbf{A}) = \int_{\mathbb{R}^{n\times\ell}} f(X)\mathrm{d}\,\lambda(X).$$

- $\mathrm{LI}_{n,\ell}$: $n\times\ell$ integer matrix, linearly indepependent columns
- $\mathrm{PR}_{n,\ell}$: $n\times\ell$ integer matrix, primitive†

*There is an error in the original proof of Rogers, noticed and fixed in [Kim24].

†Can be extended to form an integer basis of $\mathbb{Z}^n$.

HSVP: formal result

Theorem (HSVP)

There is a randomized algorithm that for every $n \geqslant 2$ *and* $\beta \in (0, \frac{n-2}{2n})$ *and on a fraction at least* $1 - 2^{1-n/2}(1+o(1)) - (2e\beta)^{-n/2}O(n^{3/2})$ *of random lattices* $\mathcal{L}$ *according to* μ_n*, outputs in time* $e^{o(n)}(\beta e^{1-\beta})^{-n/2} + 2^{n/2+o(n)}$ *and space* $2^{n/2+o(n)}$ *a nonzero vector of* $\mathcal{L}$ *of length at most* $s_{1/2}\sqrt{\frac{n\beta}{\pi}}$ *with probability at least* $1/2$*, where* $s_{1/2} = (6+4\sqrt{2})^{1/n} = 1 + o(1)$

Corollary (HSVP)

There is a randomized algorithm that for every $n \geqslant 2$ *and* $\beta \in [\frac{1}{2e}, 0.2320]$*, solves* $(1+o(1))\sqrt{\frac{n\beta}{\pi}}$*-HSVP in time* $e^{o(n)}(\beta e^{1-\beta})^{-n/2}$ *and space* $2^{n/2+o(n)}$ *with probability at least* $1/2$ *on a fraction at least* $1 - e^{o(n)}(2e\beta)^{-n/2}$ *of random lattices according to* μ_n*.*

SVP: formal result

Theorem (γ-SVP ($\gamma > 1$))

There is a randomized algorithm that for every $n \geqslant 1$, $\gamma \in (1, 1.1230]$ and $\beta \in [\frac{1}{2e}, \frac{\gamma^2}{2e})$, solves γ-SVP in time $e^{o(n)}(\beta e^{1-\beta})^{-n/2}$ and space $2^{n/2+o(n)}$ with probability at least $1/2$ on a fraction at least $1 - O(n^{3/2})(2e\beta)^{-n/2} - O(n^{-1/2})\left(\frac{2e\beta}{\gamma^2}\right)^{n/2}$ of random lattices $\mathcal{L}$ according to μ_n.

Theorem (SVP ($\gamma = 1$))

There is a randomized algorithm that for every $n \geqslant 1$ and $\alpha > 2^{1/n}$, and on a fraction at least $1 - 2^{1-n/2}(1 + o(1)) - \frac{2\alpha^n A(n)}{(\alpha^n - 2)^2}$ of random lattices $\mathcal{L}$ according to μ_n, outputs in time $2^{\tau n + o(n)}$ and space $2^{n/2+o(n)}$ a shortest nonzero vector of $\mathcal{L}$ with probability at least $1/2$, where $\tau = \frac{1}{2} + \frac{\alpha^2}{4e \ln 2}$ and $A(n) = 1 + 2^{1-n}(1 + o(1))$.

SVP: almost all lattices

Corollary (γ-SVP on almost all lattices ($\gamma > 1$))

There is a randomized algorithm that for every $\gamma \in (\frac{1}{0.99}, \sqrt{2e\beta_{\max}}]$ *and* $n \geqslant 1$ *solves* γ*-SVP in time* $\left(0.1821\gamma^2 e^{1-0.1821\gamma^2}\right)^{-n/2} e^{o(n)}$ *and space* $2^{n/2+o(n)}$ *with probability at least* $1/2$ *on a fraction at least* $1 - 0.99^{n/2+o(n)}$ *of random lattices* $\mathcal{L}$ *according to* μ_n.

Corollary (SVP on almost all lattices ($\gamma = 1$))

There is a randomized algorithm that for every $n \geqslant 70$*, solves SVP in time* $2^{0.635n+o(n)}$ *with probability at least* $1/2$ *on a fraction at least* $1 - 0.99^{n/2}$ *of random lattices* $\mathcal{L}$ *according to* μ_n.

SVP: most lattices

Corollary (γ-SVP on most lattices ($\gamma > 1$))

For every $k > 0$, there is a randomized algorithm that for every $\gamma \in (1, \sqrt{2e\beta_{\max}}]$ and $n \geqslant -\frac{k}{\ln \gamma} W_{-1}(-\frac{\ln \gamma}{k})$ solves γ-SVP in space $2^{n/2+o(n)}$ and time $e^{o(n)}(\frac{2}{\gamma^2}e^{\gamma^2/2e})^{n/2}$ with probability at least $1/2$ on a fraction at least $1 - O(n^{-k})$ of random lattices $\mathcal{L}$ according to μ_n.

Corollary (SVP on most lattices ($\gamma = 1$))

There is a randomized algorithm that for every $k \in \mathbb{N}$ and $n \geqslant 1$, solves SVP in time $2^{(\frac{1}{2}+\frac{1}{4e\ln 2})n+o(n)} = 2^{0.63269n+o(n)}$ with probability at least $1/2$ on a fraction at least $1 - 2n^{-k}(1 + o(1))$ of random lattices $\mathcal{L}$ according to μ_n.