A universal ordinary differential equation

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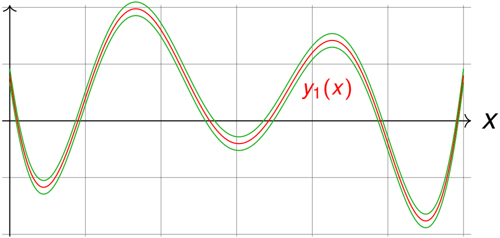
Universal differential algebraic equation (Rubel)



Theorem (Rubel, 1981)

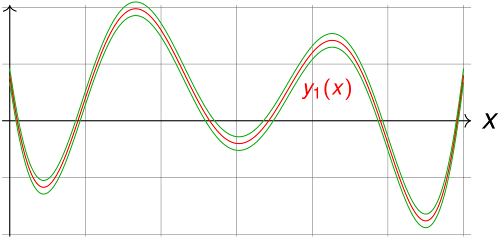
For any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$\begin{aligned} 3y'^4 y'' y''''^2 & -4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ & -12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 \qquad = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Universal differential algebraic equation (Rubel)



Theorem (Rubel, 1981)

There exists a **fixed** k and nontrivial polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$p(y, y', \ldots, y^{(k)}) = 0$$

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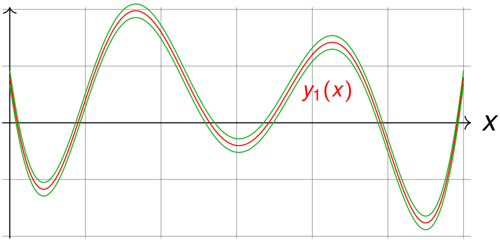

Open Problem

Can we have unicity of the solution with initial conditions ?

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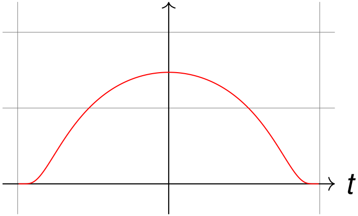
Rubel's ("disappointing") proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

$$\text{It satisfies} \quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$$



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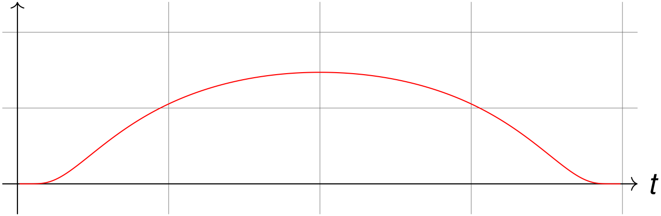

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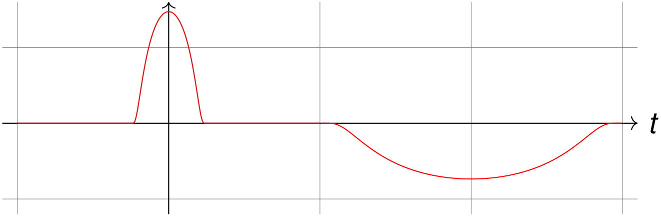

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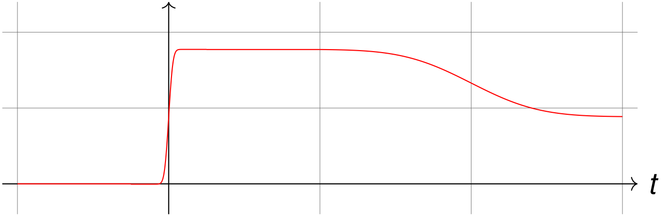
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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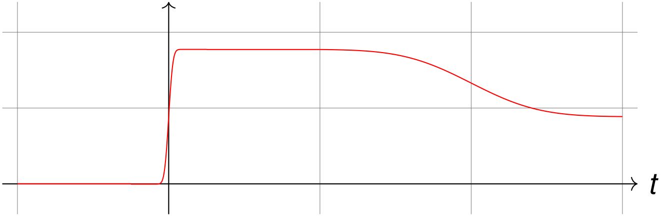

Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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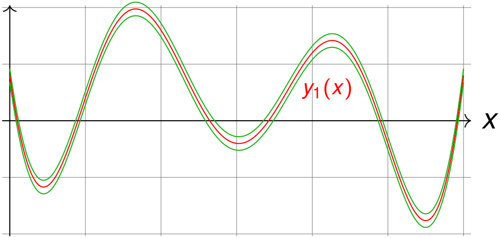
- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Universal explicit ordinary differential equation



Main result

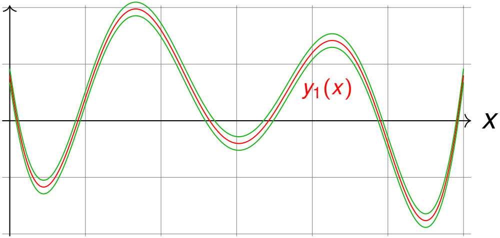
There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Universal explicit ordinary differential equation



Notes :

- **system** of ODEs,
- y must be analytic,
- we need $d \approx 300$.

Main result

There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

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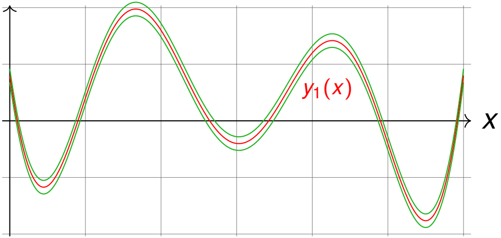
Universal DAE, again but better



Corollary of main result

There exists a **fixed** k and nontrivial polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

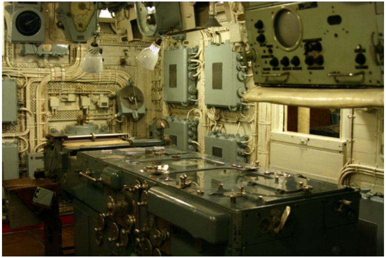
has a **unique analytic solution** $y : \mathbb{R} \to \mathbb{R}$ and $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Some motivation

Polynomial ODEs correspond to analog computers :

Differential Analyser



British Navy mecanical computer

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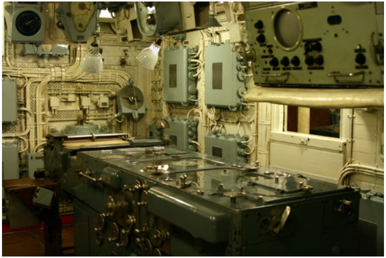
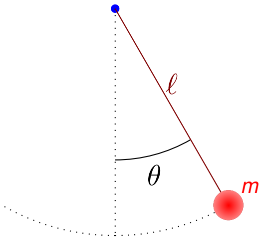


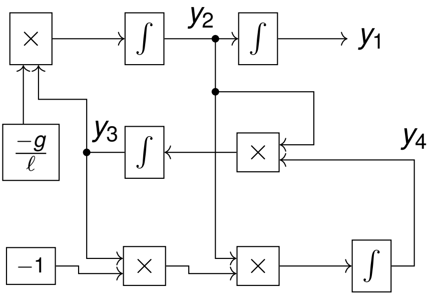
British Navy mecanical computer

- They are **equivalent** to Turing machines !
- One can **characterize P** with pODEs (ICALP 2016)

Take away : polynomial ODEs are a natural programming language.

Example of differential equation



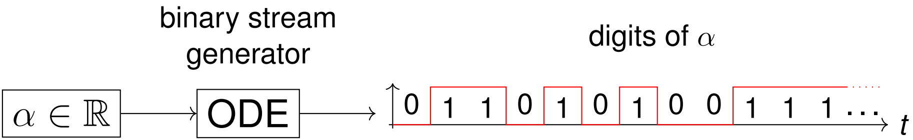


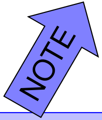
General Purpose Analog Computer (GPAC)
Shannon's model of the Differential Analyser

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{\ell} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

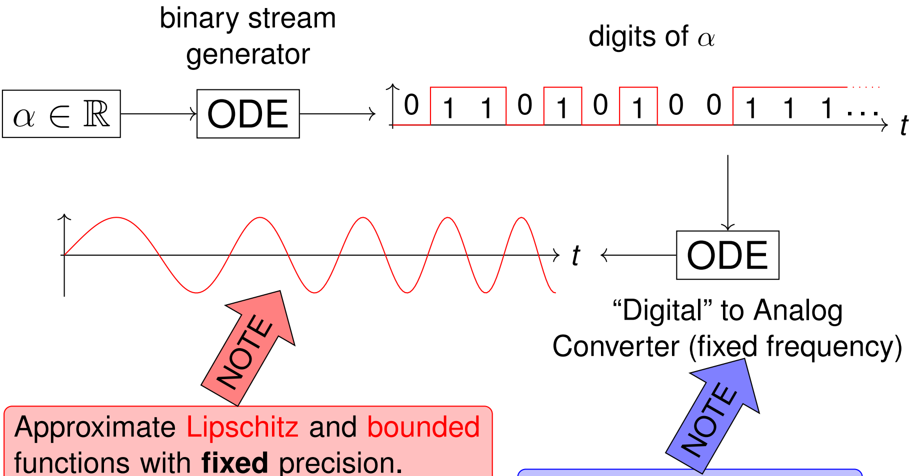
A simplified proof





This is the ideal curve, the real one is an approximation of it.

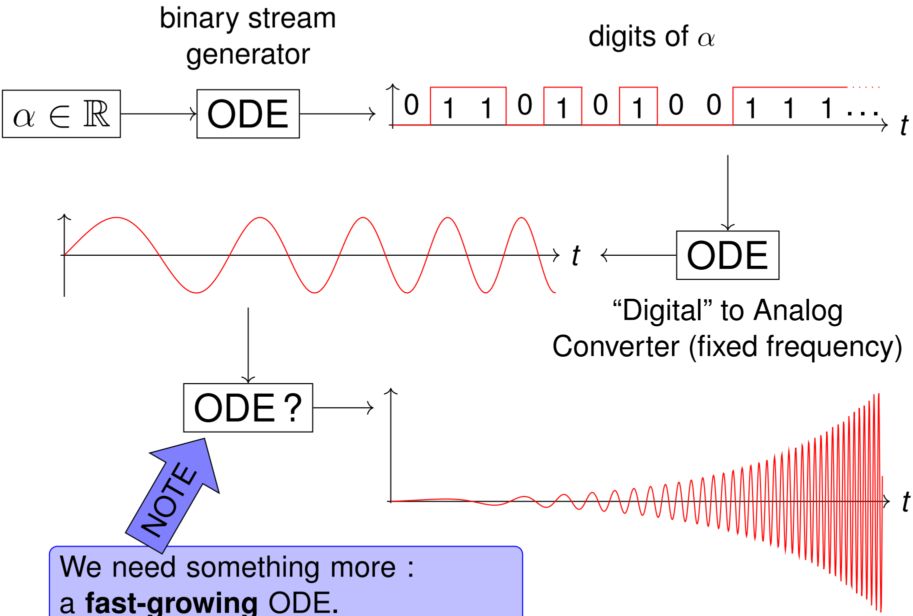
A simplified proof



Approximate Lipschitz and bounded functions with **fixed** precision.

That's the trickiest part.

A simplified proof



We need something more :
a **fast-growing** ODE.

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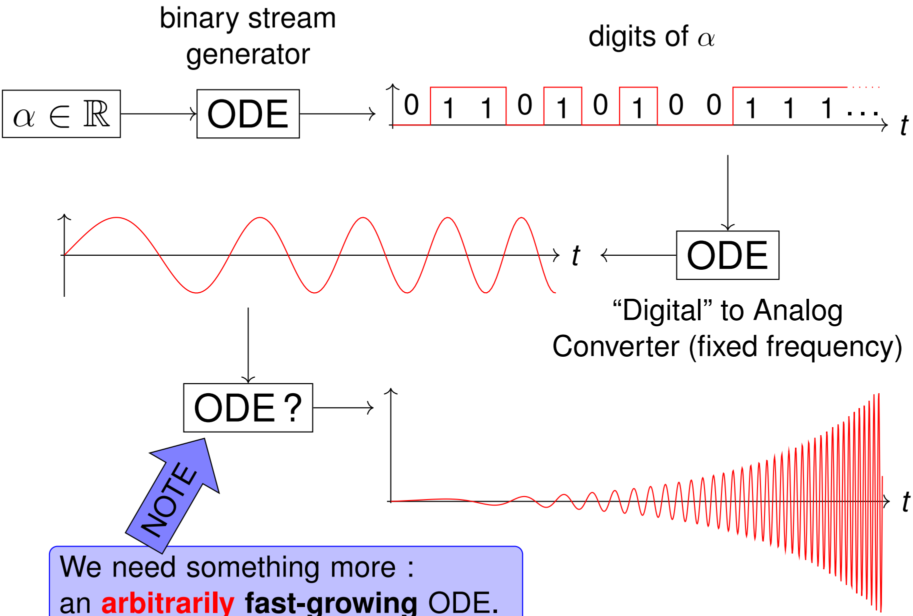

We need something more :
an **arbitrarily** **fast-growing** ODE.

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \qquad \rightsquigarrow \qquad y_1(t) = \exp(t)$$

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Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

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$e_n(t) = \exp(\cdots\exp(t)\cdots)$ (n compositions)

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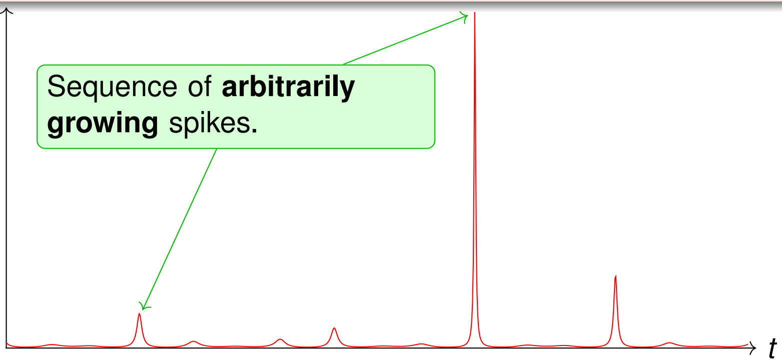
Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2-\cos(t)-\cos(\alpha t)}$$



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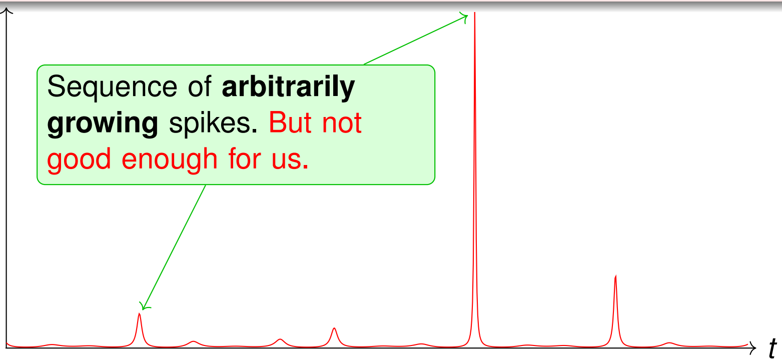

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Theorem (In the paper)

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

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satisfies

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Note : both results require α to be **transcendental**. Conjecture still open for **rational** coefficients.

Proof gem : iteration with differential equations

Goal

Iterate f with a GPAC : $y(n) \approx f^{[n]}([x])$

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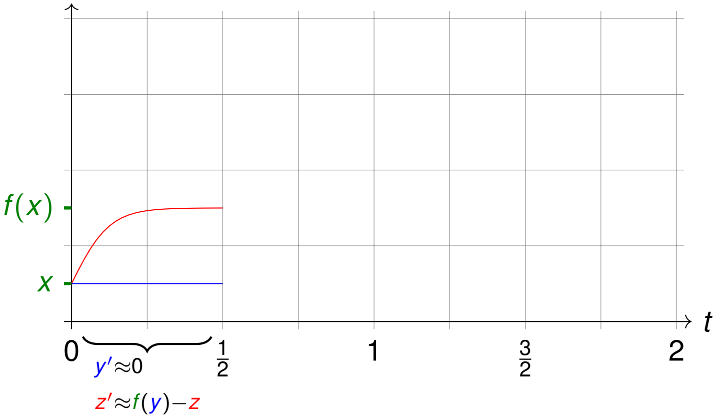

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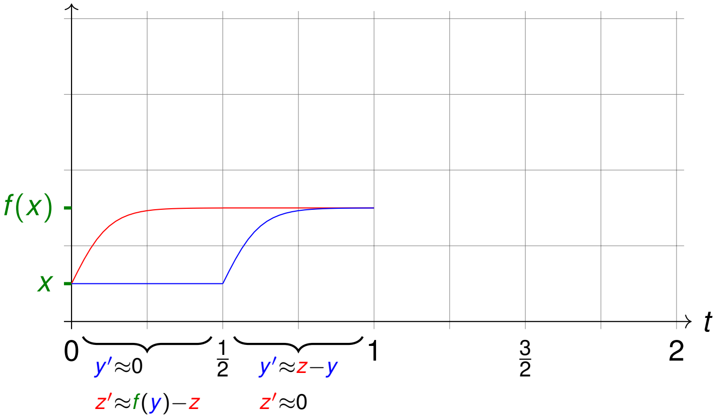

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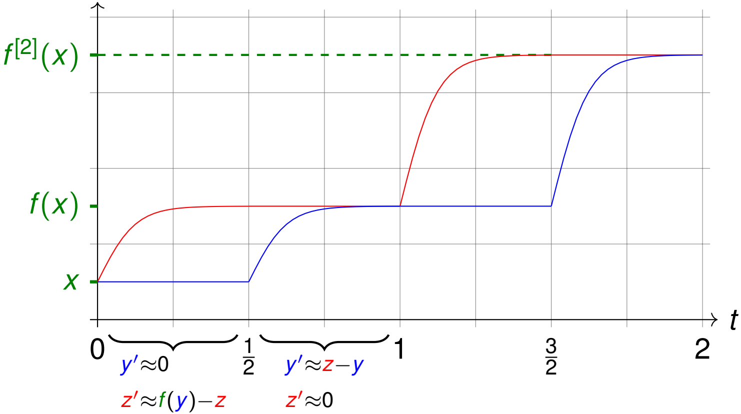

A computability question

Main result (reminder)

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Question : is α computable from f and ε ?

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Question : is α computable from f and ε ? Yes [1]

1. With the right notion of computability.

Conclusion

This paper

positive answer to Rubel's open problem

Take home

ODE is a simple, nice and fun programming language

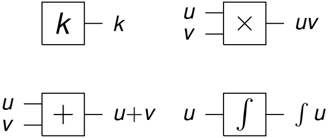
Possible development

Each universal ODE defines a map :

$$(f, \varepsilon) \in C^0 \times C^0 \mapsto \alpha \in \mathbb{R}$$

Kolmogorov-like complexity for continuous functions ?

Polynomial Differential Equations



General Purpose
Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks :

- chemical
- enzymatic

polynomial differential equations :

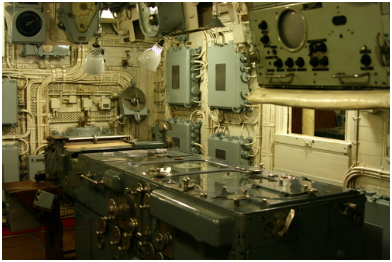
$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable ($+, \times, \circ, /$, ED)
- No closed-form solution

Digital vs analog computers

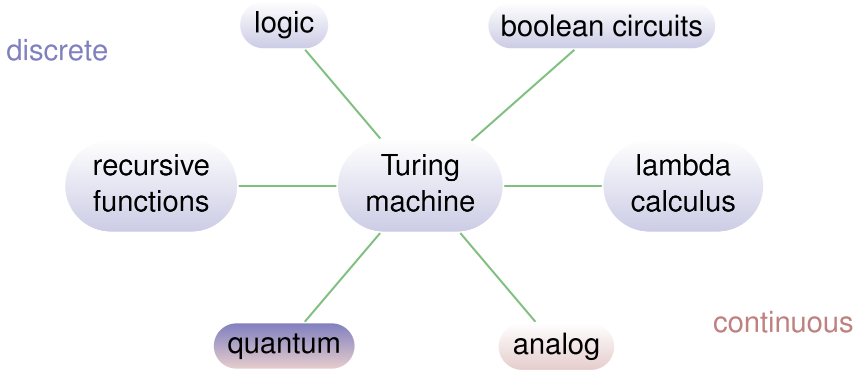
Digital vs analog computers

VS



Church Thesis

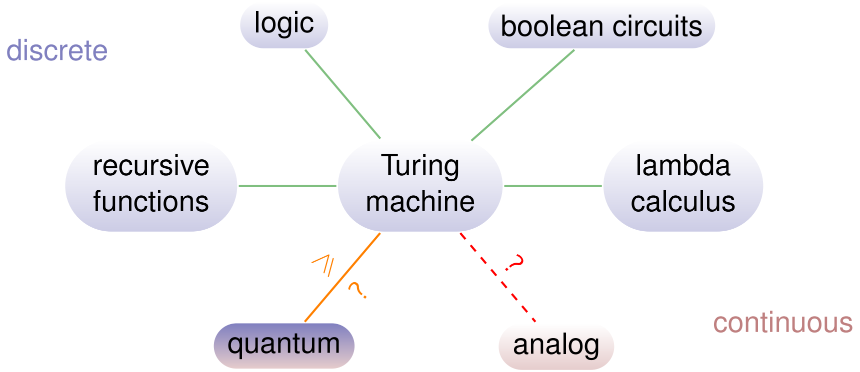
Computability



Church Thesis

All reasonable models of computation are equivalent.

Complexity



Effective Church Thesis

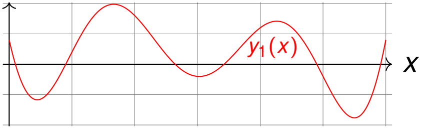
All reasonable models of computation are equivalent for complexity.

Computing with the GPAC

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$f(x) = y_1(x)$



Shannon's notion

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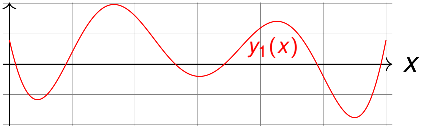

Shannon's notion

sin, cos, exp, log, ...

Strictly weaker than Turing machines [Shannon, 1941]

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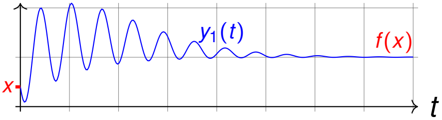
Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_{\geqslant 0} \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$



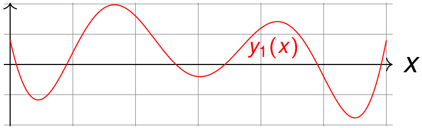
Modern notion

Computing with the GPAC

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



Shannon's notion

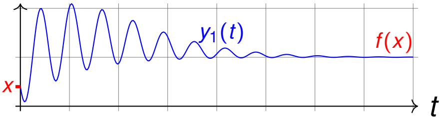
sin, cos, exp, log, ...

Strictly weaker than Turing machines [Shannon, 1941]

Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_{\geqslant 0} \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$



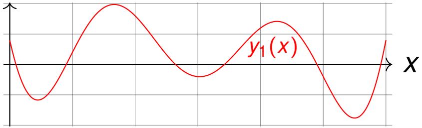
Modern notion

sin, cos, exp, log, Γ, ζ, ...

Turing powerful [Bournez et al., 2007]

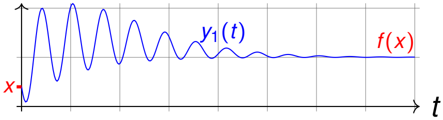
Universal differential equations

Generable functions



subclass of analytic functions

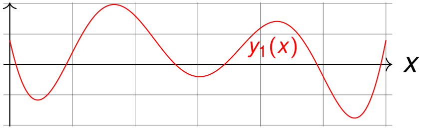
Computable functions



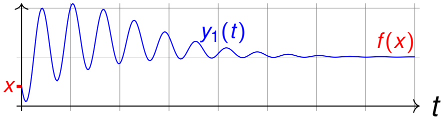
any computable function

Universal differential equations

Generable functions

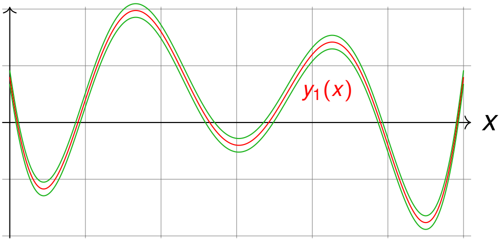


Computable functions



subclass of analytic functions

any computable function



A new notion of computability

Almost-Theorem

$f : [0,1] \to \mathbb{R}$ is **computable** if and only if there exists $\tau > 1$, $y_0 \in \mathbb{R}^d$ and p polynomial such that

$$y'(0) = y_0, \qquad y'(t) = p(y(t))$$

satisfies

$$|f(x) - y(x + n\tau)| \leqslant 2^{-n}, \qquad \forall x \in [0,1], \forall n \in \mathbb{N}$$