On the Decidability of Reachability in Linear Time-Invariant Systems

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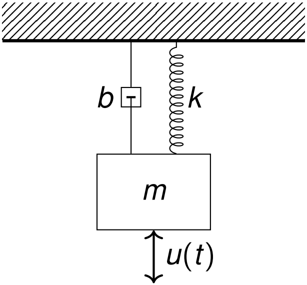
16 april 2019

Example : mass-spring-damper system



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State : $X = z \in \mathbb{R}$

Equation of motion :

$$mz'' = -kz - bz' + mg + u$$

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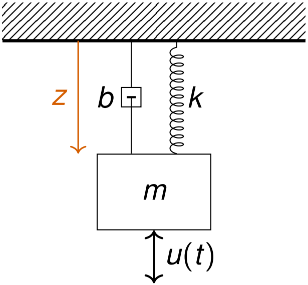

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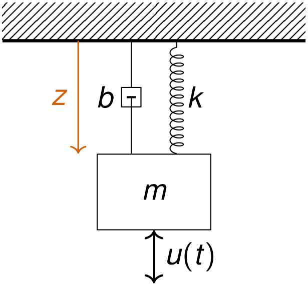

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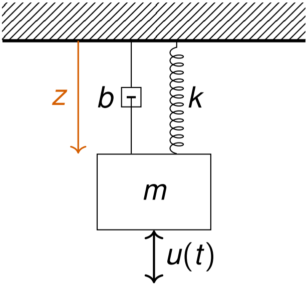

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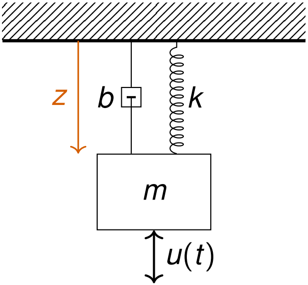

Model with external input $u(t)$
$\rightarrow$ Linear time invariant system

$$X' = AX + Bu$$

with some constraints on u.

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Linear dynamical systems

Discrete case

$$x(n+1) = Ax(n)$$

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
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Continuous case

$$x'(t) = Ax(t)$$

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Typical questions

- reachability
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The problem

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- a source $s \in \mathbb{Q}^d$,
- a target $t \in \mathbb{Q}^d$,
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- a set of controls $U \subseteq \mathbb{R}^d$,

decide if $\exists T \in \mathbb{N}, u_0, \ldots, u_{T-1} \in U$ such that $x_T = t$ where

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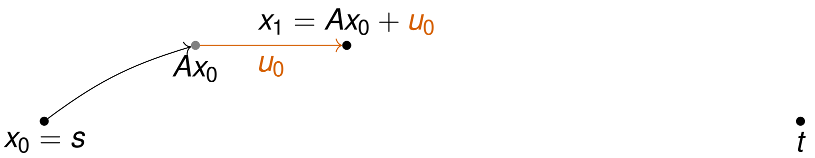

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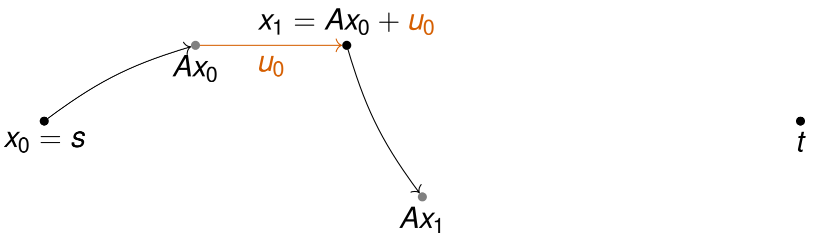

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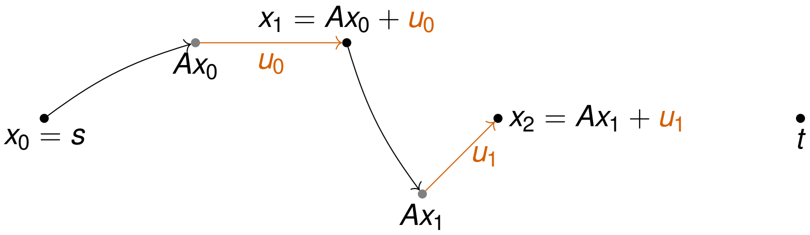

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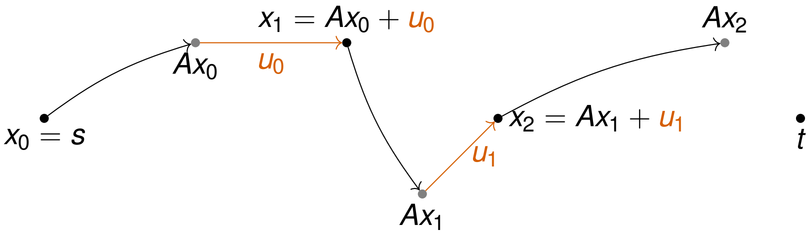

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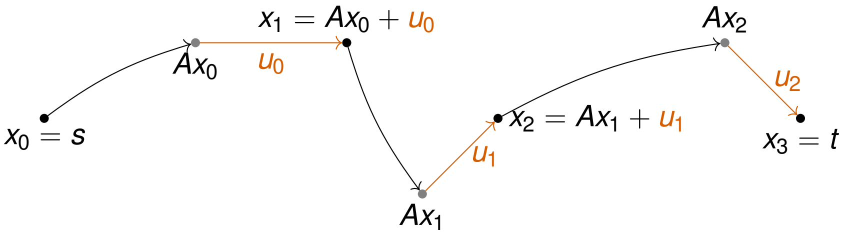

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Almost no exact results for other classes of U in particular when U is bounded (which is the most natural case).

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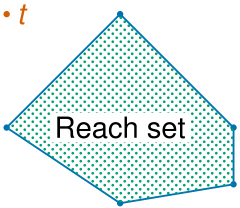

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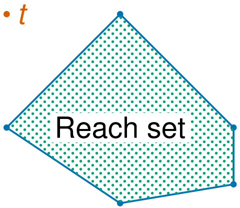

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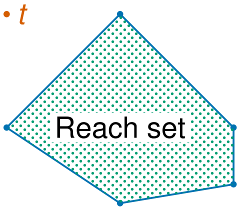

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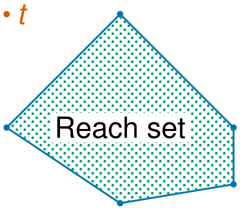

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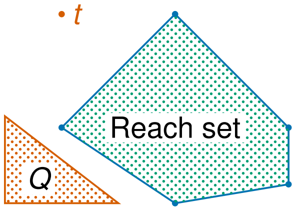
Remark : in fact we can decide reachability to a convex polytope Q.



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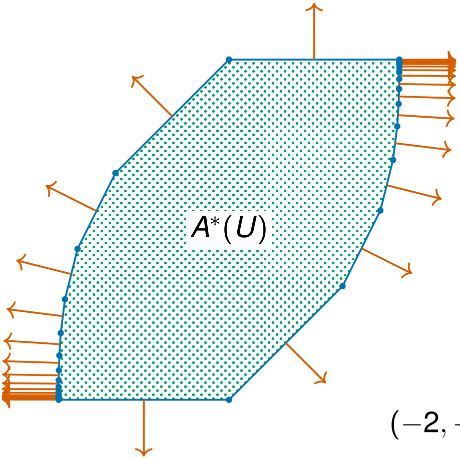
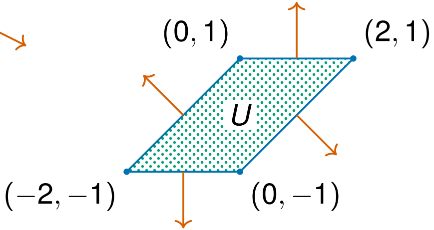
Why is this problem hard

The reachable set $A^*(U)$ can have **infinitely** many faces.



$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$



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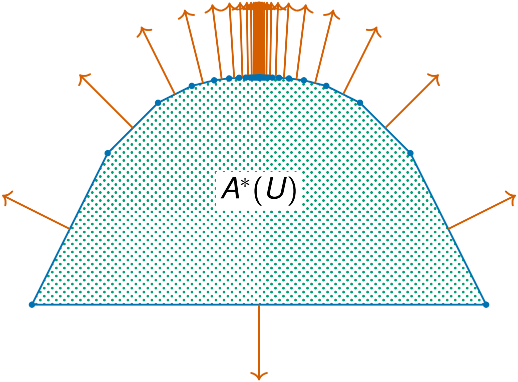
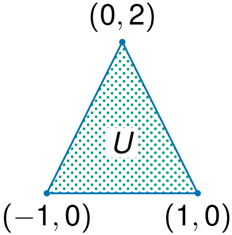
The reachable set $A^*(U)$ can have **faces of lower dimension** : the "top" extreme point does not belong to any facet.



$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$



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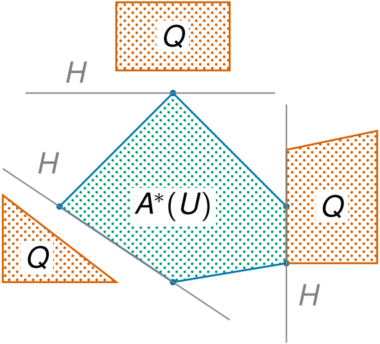

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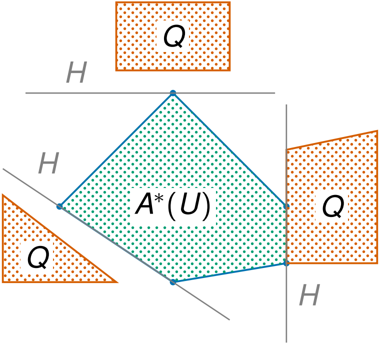

Further difficulty : a separating hyperplane may not be supported by a facet of either $A^*(U)$ or Q.

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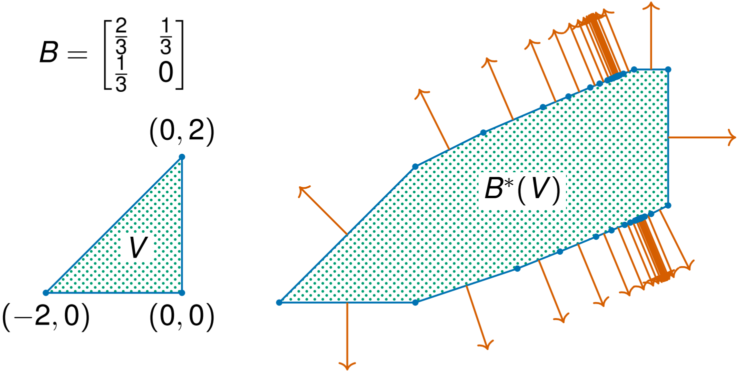
$$B = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$



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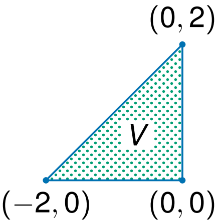

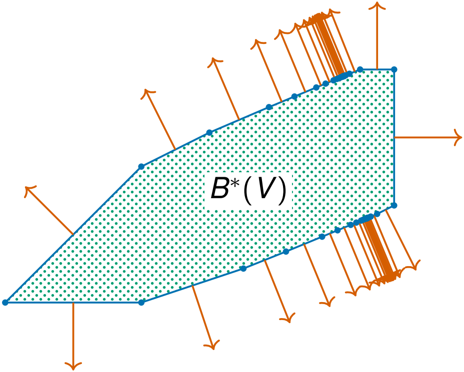

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Theorem (Non-reachable instances)

There is a separating hyperplane with algebraic coefficients.

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$$A = \frac{1}{2}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \qquad U = [0,1] \times \{0\}.$$

Decidability of $t \leqslant \sum_{n=0}^{\infty} \max(0, 2^{-n}\cos(n\theta))$ unknown.

Future work : continuous case $x'(t) = Ax(t) + u(t)$

Backup slides