A universal differential equation

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Joint work with Olivier Bournez and Daniel Graça

14 June 2019
What is a computer?
What is a computer?
What is a computer?

VS

[Image of a modern laptop] vs [Image of an old mechanical computer]
All **reasonable** models of computation are equivalent.
Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.
Polynomial Differential Equations

General Purpose Analog Computer

Newton mechanics

Reaction networks:
- chemical
- enzymatic

polynomial differential equations:
\[ \begin{align*}
y(0) &= y_0 \\
y'(t) &= p(y(t))
\end{align*} \]

- Rich class
- Stable (+,×,○,/,ED)
- No closed-form solution
Example of dynamical system

\[ \ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \]
Example of dynamical system

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Historical remark: the word "analog"
Example of dynamical system

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Historical remark: the word "analog". The pendulum and the circuit have the same equation. One can study one using the other by analogy.
Example of dynamical system

\[ \ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \]

Historical remark: the word “analog”

The pendulum and the circuit have the same equation. One can study one using the other by analogy.
Computing with differential equations

Generable functions

\[
\begin{cases}
    y(0) = y_0 \\
y'(x) = p(y(x))
\end{cases} \quad x \in \mathbb{R}
\]

\[f(x) = y_1(x)\]

Shannon’s notion

\[f(x) = \lim_{t \to \infty} y_1(t)\]
Computing with differential equations

**Generable functions**

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\begin{align*}
    y(0) &= y_0 \\
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\end{align*}
\]

\[x \in \mathbb{R}\]

\[f(x) = y_1(x)\]

**Shannon’s notion**

sin, cos, exp, log, ...

**Strictly weaker than Turing machines** [Shannon, 1941]
Computing with differential equations

Generable functions

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\begin{align*}
    \begin{cases}
    y(0) &= y_0 \\
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    & \quad x \in \mathbb{R}
\end{align*}
\]

\[f(x) = y_1(x)\]

Shannon’s notion
sin, cos, exp, log, ...

Strictly weaker than Turing machines [Shannon, 1941]

Computable

\[
\begin{align*}
    \begin{cases}
    y(0) &= q(x) \\
    y'(t) &= p(y(t))
    \end{cases}
    & \quad x \in \mathbb{R}
\end{align*}
\]

\[f(x) = \lim_{t \to \infty} y_1(t)\]

Modern notion
Computing with differential equations

Generable functions

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\begin{cases}
  y(0) = y_0 \\
y'(x) = p(y(x))
\end{cases} \quad x \in \mathbb{R}
\]

\[f(x) = y_1(x)\]

Shannon’s notion

\[\sin, \cos, \exp, \log, \ldots\]

Strictly weaker than Turing machines [Shannon, 1941]

Computable

\[
\begin{cases}
  y(0) = q(x) \\
y'(t) = p(y(t))
\end{cases} \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+
\]

\[f(x) = \lim_{t \to \infty} y_1(t)\]

Modern notion

\[\sin, \cos, \exp, \log, \Gamma, \zeta, \ldots\]

Turing powerful

[Bournez et al., 2007]
Equivalence with computable analysis

Definition (Bournez et al, 2007)

\(f\) computable by GPAC if \(\exists p\) polynomial such that \(\forall x \in [a, b]\)

\(y(0) = (x, 0, \ldots, 0)\quad y'(t) = p(y(t))\)

satisfies \(|f(x) - y_1(t)| \leq y_2(t)\) and \(y_2(t) \xrightarrow{t \to \infty} 0\).

\(y_1(t) \xrightarrow{t \to \infty} f(x)\)

\(y_2(t) = \text{error bound}\)
Equivalence with computable analysis

Definition (Bournez et al, 2007)

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y(0) = (x, 0, \ldots, 0) \quad y'(t) = p(y(t))
\]

satisfies \( |f(x) - y_1(t)| \leq y_2(t) \) et \( y_2(t) \xrightarrow{t \to \infty} 0. \)

![Graph showing the function y1(t) and f(x) approaching y2(t) as t approaches infinity.]

Theorem (Bournez et al, 2007)

\( f : [a, b] \to \mathbb{R} \) computable \( ^1 \) \( \iff \) \( f \) computable by GPAC

---

1. In Computable Analysis, a standard model over reals built from Turing machines.
Equivalence with computable analysis

Definition (Bournez et al, 2007)

\[ f \text{ computable by GPAC if } \exists p \text{ polynomial such that } \forall x \in [a, b] \]
\[ y(0) = (x, 0, \ldots, 0) \quad y'(t) = p(y(t)) \]
satisfies \(|f(x) - y_1(t)| \leq y_2(t) \text{ et } y_2(t) \xrightarrow{t \to \infty} 0.\]

Theorem (Bournez et al, 2007)

\[ f : [a, b] \to \mathbb{R} \text{ computable}^1 \iff f \text{ computable by GPAC} \]

---

1. In Computable Analysis, a standard model over reals built from Turing machines.
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Theorem (Rubel, 1981)

For any continuous functions $f$ and $\varepsilon$, there exists $y : \mathbb{R} \rightarrow \mathbb{R}$ solution to

$$3y^4y''y^{''''^2} - 4y'^4y''^2y^{'''''} + 6y'^3y''y^{'''}y^{''''} + 24y'^2y''^4y^{''''}$$

$$-12y'^3y''y^{'''}^3 - 29y'^2y''^3y^{''^2} + 12y''^7 = 0$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Universal differential algebraic equation (DAE)

Theorem (Rubel, 1981)

There exists a fixed polynomial \( p \) and \( k \in \mathbb{N} \) such that for any continuous functions \( f \) and \( \varepsilon \), there exists a solution \( y : \mathbb{R} \to \mathbb{R} \) to

\[
p(y, y', \ldots, y^{(k)}) = 0
\]

such that \( \forall t \in \mathbb{R}, \)

\[
|y(t) - f(t)| \leq \varepsilon(t).
\]
Theorem (Rubel, 1981)

There exists a fixed polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

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such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \varepsilon(t).$$

Problem: this is «weak» result.
The problem with Rubel’s DAE

The solution $y$ is not unique, even with added initial conditions:

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, \ y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel’s proof to work!
The problem with Rubel’s DAE

The solution $y$ is not unique, even with added initial conditions:

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In fact, this is fundamental for Rubel’s proof to work!

- Rubel’s statement: this DAE is universal
- More realistic interpretation: this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?

Note: explicit polynomial ODE $\Rightarrow$ unique solution
There exists a fixed (vector of) polynomial \( p \) such that for any continuous functions \( f \) and \( \varepsilon \), there exists \( \alpha \in \mathbb{R}^d \) such that

\[
y(0) = \alpha, \quad y'(t) = p(y(t))
\]

has a unique solution \( y : \mathbb{R} \to \mathbb{R}^d \) and \( \forall t \in \mathbb{R} \),

\[
|y_1(t) - f(t)| \leq \varepsilon(t).
\]
Universal initial value problem (IVP)

Notes:
- system of ODEs,
- $y$ is analytic,
- we need $d \approx 300$.

Theorem

There exists a fixed (vector of) polynomial $p$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha \in \mathbb{R}^d$ such that

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has a unique solution $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$

Remark: $\alpha$ is usually transcendental, but computable from $f$ and $\varepsilon$. 
Take \( f(t) = e^{\frac{-1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise.

It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0\).
Rubel’s proof in one slide

- Take \( f(t) = e^{\frac{-1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise.

  It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0\).

- For any \( a, b, c \in \mathbb{R} \), \( y(t) = cf(at + b) \) satisfies

\[
3y'^4y''y''''^2 - 4y'^4y''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y'''^3y''^2 + 12y''^7 = 0
\]
Rubel’s proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.
  It satisfies $(1 - t^2)^2 f''(t) + 2tf'(t) = 0$.

- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at + b)$ satisfies
  
  $$
  3y^{(4)}y''y'''^2 - 4y^{(4)}y''^2y'' + 6y^{(3)}y''^2y''' + 24y^{(2)}y''^4y'' + 12y^{(3)}y''^3 - 29y^{(3)}y'' + 12y^{(7)} = 0
  $$

- Can glue together arbitrary many such pieces
Rubel’s proof in one slide

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- Can glue together arbitrary many such pieces

- Can arrange so that \( \int f \) is solution: piecewise pseudo-linear
Take \( f(t) = e^{\frac{-1}{1-t^2}} \) for \(-1 < t < 1\) and \( f(t) = 0 \) otherwise. It satisfies \((1 - t^2)^2 f''(t) + 2tf'(t) = 0.\)

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\]

Can glue together arbitrary many such pieces

Can arrange so that \( \int f \) is solution: piecewise pseudo-linear

Conclusion: Rubel’s equation allows any piecewise pseudo-linear functions, and those are dense in \( C^0 \)
Theorem

There exists a fixed polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, \ y^{(k)}(0) = \alpha_k$$

has a unique analytic solution and this solution satisfies such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by programming with ODEs.
### Generable functions (total, univariate)

#### Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is *generable* if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

#### Types

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coeff. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d$, $y : \mathbb{R} \rightarrow \mathbb{R}^d$

Note: existence and unicity of $y$ by Cauchy-Lipschitz theorem.
### Generable functions (total, univariate)

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satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example:** $f(x) = x$  ► identity

$$y(0) = 0, \quad y' = 1 \quad \leadsto \quad y(x) = x$$

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y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Example:** \( f(x) = x^2 \)  \[ \rightarrow \] squaring

\[
y_1(0) = 0, \quad y'_1 = 2y_2 \quad \Rightarrow \quad y_1(x) = x^2
\]

\[
y_2(0) = 0, \quad y'_2 = 1 \quad \Rightarrow \quad y_2(x) = x
\]

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\( f : \mathbb{R} \to \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

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Example: \( f(x) = x^n \)

- \( y_1(0) = 0, \quad y_1' = ny_2 \quad \leadsto \quad y_1(x) = x^n \)
- \( y_2(0) = 0, \quad y_2' = (n - 1)y_3 \quad \leadsto \quad y_2(x) = x^{n-1} \)
- \( \ldots \)
- \( y_n(0) = 0, \quad y_n = 1 \quad \leadsto \quad y_n(x) = x \)
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

\[ y(0) = y_0, \quad y'(x) = p(y(x)) \]

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Example:** $f(x) = \exp(x)$  

\[ y(0) = 1, \quad y' = y \quad \Rightarrow \quad y(x) = \exp(x) \]

**Types**

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- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$
Generable functions (total, univariate)

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\( f : \mathbb{R} \rightarrow \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

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**Example:** \( f(x) = \sin(x) \) or \( f(x) = \cos(x) \)

\[
y_1(0) = 0, \quad y'_1 = y_2 \quad \sim \quad y_1(x) = \sin(x)
\]
\[
y_2(0) = 1, \quad y'_2 = -y_1 \quad \sim \quad y_2(x) = \cos(x)
\]
**Generable functions (total, univariate)**

**Definition**

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

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**Example** : $f(x) = \tanh(x)$

- $\tanh(x)$

$$y(0) = 0, \quad y' = 1 - y^2 \quad \sim \quad y(x) = \tanh(x)$$
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

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y(0) = y_0, \quad y'(x) = p(y(x))
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satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

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- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d \)

**Example**

\( f(x) = \frac{1}{1+x^2} \)  

\[
f'(x) = \frac{-2x}{(1+x^2)^2} = -2xf(x)^2
\]

\[
y_1(0) = 1, \quad y_1' = -2y_2y_1^2 \quad \leadsto \quad y_1(x) = \frac{1}{1+x^2}
\]

\[
y_2(0) = 0, \quad y_2' = 1 \quad \leadsto \quad y_2(x) = x
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Generable functions (total, univariate)

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\( f : \mathbb{R} \to \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

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**Example:** \( f = g \pm h \quad \text{sum/difference} \)

\[
(g \pm h)' = g' \pm h'
\]

**assume:**

\[
\begin{align*}
z(0) &= z_0, & z' &= p(z) & \sim & z_1 &= g \\
w(0) &= w_0, & w' &= q(w) & \sim & w_1 &= h
\end{align*}
\]

**then:**

\[
\begin{align*}
y(0) &= z_{0,1} + w_{0,1}, & y' &= p_1(z) \pm q_1(w) & \sim & y &= z_1 \pm w_1
\end{align*}
\]
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**Definition**

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**Example** : \( f = gh \)  ► product

\[
(gh)' = g' h + gh'
\]

**assume** :

\[
\begin{align*}
z(0) &= z_0, & z' &= p(z) & \Rightarrow z_1 &= g \\
w(0) &= w_0, & w' &= q(w) & \Rightarrow w_1 &= h
\end{align*}
\]

**then** :

\[
\begin{align*}
y(0) &= z_{0,1} w_{0,1}, & y' &= p_1(z)w_1 + z_1 q_1(w) & \Rightarrow y &= z_1 w_1
\end{align*}
\]
Generable functions (total, univariate)

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\( f : \mathbb{R} \to \mathbb{R} \) is **generable** if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

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**Example**

\( f = \frac{1}{g} \)  ➤ inverse

\[
f' = \frac{-g'}{g^2} = -g' f^2
\]

**assume**:

\[
z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g
\]

**then**:

\[
y(0) = \frac{1}{z_{0,1}}, \quad y' = -p_1(z)y^2 \quad \leadsto \quad y = \frac{1}{z_1}
\]
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- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$ : polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

**Example** : $f = \int g$

*assume* :

$$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$$

*then* :

$$y(0) = 0, \quad y' = z_1 \quad \leadsto \quad y = \int z_1$$
Generable functions (total, univariate)

Definition

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]
satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

Types

- \( d \in \mathbb{N} \): dimension
- \( \mathbb{Q} \subseteq K \subseteq \mathbb{R} \): field
- \( p \in K^d[\mathbb{R}^n] \): polynomial vector (coef. in \( K \))
- \( y_0 \in K^d, y : \mathbb{R} \rightarrow \mathbb{R}^d \)

Example: \( f = g' \)  

\[
f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'
\]

Assume:

\[
z(0) = z_0, \quad z' = p(z) \quad \Rightarrow \quad z_1 = g
\]

Then:

\[
y(0) = p_1(z_0), \quad y' = \nabla p_1(z) \cdot p(z) \quad \Rightarrow \quad y = z_1''
\]
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is *generable* if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$

**Example**: $f = g \circ h$

$$z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

*assume*:

- $z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$
- $w(0) = w_0, \quad w' = q(w) \quad \leadsto \quad w_1 = h$

*then*:

- $y(0) = z(w_0), \quad y' = p(y)z_1 \quad \leadsto \quad y = z \circ h$
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Types**

- \( d \in \mathbb{N} \) : dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \) : field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] \) : polynomial vector (coef. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d \)

**Example** : \( f = g \circ h \)  

\[
(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'
\]

**assume** :

\[
\begin{align*}
z(0) &= z_0, & z' &= p(z) & \leadsto & z_1 &= g \\
w(0) &= w_0, & w' &= q(w) & \leadsto & w_1 &= h
\end{align*}
\]

**then** :

\[
\begin{align*}
y(0) &= z(w_0), & y' &= p(y)z_1 & \leadsto & y &= z \circ h
\end{align*}
\]

Is this coefficient in \( \mathbb{K} \) ?
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$

**Example:** $f = g \circ h$  

- composition

\[
(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'
\]

**assume:**

\[
\begin{align*}
z(0) &= z_0, & z' &= p(z) & \leadsto & z_1 &= g \\
w(0) &= w_0, & w' &= q(w) & \leadsto & w_1 &= h
\end{align*}
\]

**then:**

\[
\begin{align*}
y(0) &= z(w_0), & y' &= p(y)z_1 & \leadsto & y &= z \circ h
\end{align*}
\]

Is this coefficient in $\mathbb{K}$? Fields with this property are called **generable**.
Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is **generable** if there exists $d, p$ and $y_0$ such that the solution $y$ to

\[ y(0) = y_0, \quad y'(x) = p(y(x)) \]

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d$, $y : \mathbb{R} \to \mathbb{R}^d$

**Example**:

\[ f' = \tanh \circ f \]

**Non-polynomial differential equation**

\[ f'' = (\tanh' \circ f)f' = (1 - (\tanh \circ f)^2)f' \]

\[
\begin{align*}
y_1(0) &= f(0), \\
y_2(0) &= \tanh(f(0)), \\
y_1'(0) &= y_2, \\
y_1'(x) &= f(x) \\
y_2'(x) &= (1 - y_2^2)y_2 \\
y_2(x) &= \tanh(f(x))
\end{align*}
\]
Generable functions (total, univariate)

**Definition**

\( f : \mathbb{R} \to \mathbb{R} \) is generable if there exists \( d, p \) and \( y_0 \) such that the solution \( y \) to

\[
y(0) = y_0, \quad y'(x) = p(y(x))
\]

satisfies \( f(x) = y_1(x) \) for all \( x \in \mathbb{R} \).

**Example:** \( f(0) = f_0, f' = g \circ f \)

**Initial Value Problem (IVP)**

\[
f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'
\]

**assume:**

\[
z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g
\]

**then:**

\[
y(0) = p_1(z_0), \quad y' = \nabla p_1(z) \cdot p(z) \quad \leadsto \quad y = z_1''
\]

**Types**

- \( d \in \mathbb{N} \): dimension
- \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R} \): field
- \( p \in \mathbb{K}^d[\mathbb{R}^n] \): polynomial vector (coeff. in \( \mathbb{K} \))
- \( y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d \)
Nice theory for the class of total and univariate *generable* functions:

- analytic
- contains polynomials, \( \sin, \cos, \tanh, \exp \)
- stable under \( \pm, \times, /, \circ \) and Initial Value Problems (IVP)
- technicality on the field \( \mathbb{K} \) of coefficients for stability under \( \circ \)
- solutions to polynomial ODEs form a **very large class**
Generable functions: a first summary

Nice theory for the class of total and univariate generable functions:

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$
- stable under $\pm$, $\times$, $/$, $\circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
- solutions to polynomial ODEs form a very large class

Limitations:

- total functions
- univariate
Generable functions (generalization)

**Definition**

\( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is generable if \( X \) is open connected and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) \) = Jacobian matrix of \( y \) at \( x \)

**Types**

- \( n \in \mathbb{N} : \) input dimension
- \( d \in \mathbb{N} : \) dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] : \)
  - polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)

**Notes :**

- Partial differential equation !
- Unicity of solution \( y \)...
- ... but not existence (ie you have to show it exists)
**Generable functions (generalization)**

**Definition**

\( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is **generable** if \( X \) is open and connected and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) \) = Jacobian matrix of \( y \) at \( x \)

**Example:** \( f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3) \)

\[
y(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3 y_2 y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sim \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}
\]

**Types**

- \( n \in \mathbb{N} \) : input dimension
- \( d \in \mathbb{N} \) : dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] \) : polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d, y : X \rightarrow \mathbb{R}^d \)
- **monomial**
Generable functions (generalization)

**Definition**

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if $X$ is open connected and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \quad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

$J_y(x) =$ Jacobian matrix of $y$ at $x$

**Types**

- $n \in \mathbb{N}$: input dimension
- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{K}^{d \times d}[\mathbb{R}^d]$: polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d$, $y : X \to \mathbb{R}^d$

**Example**

$f(x_1, x_2) = x_1 x_2^2$ ► monomial

$$y_1(0, 0) = 0, \quad \partial_{x_1} y_1 = y_3^2, \quad \partial_{x_2} y_1 = 3y_2 y_3 \quad \leadsto \quad y_1(x) = x_1 x_2^2$$

$$y_2(0, 0) = 0, \quad \partial_{x_1} y_2 = 1, \quad \partial_{x_2} y_2 = 0 \quad \leadsto \quad y_2(x) = x_1$$

$$y_3(0, 0) = 0, \quad \partial_{x_1} y_3 = 0, \quad \partial_{x_2} y_3 = 1 \quad \leadsto \quad y_3(x) = x_2$$

This is tedious!
Generable functions (generalization)

**Definition**

\( f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is generable if \( X \) is open and connected and \( \exists d, p, x_0, y_0, y \) such that

\[
y(x_0) = y_0, \quad J_y(x) = p(y(x))
\]

and \( f(x) = y_1(x) \) for all \( x \in X \).

\( J_y(x) \) = Jacobian matrix of \( y \) at \( x \)

**Types**

- \( n \in \mathbb{N} \): input dimension
- \( d \in \mathbb{N} \): dimension
- \( p \in \mathbb{K}^{d \times d}[\mathbb{R}^d] \): polynomial matrix
- \( x_0 \in \mathbb{K}^n \)
- \( y_0 \in \mathbb{K}^d \), \( y : X \rightarrow \mathbb{R}^d \)

**Last example**

\( f(x) = \frac{1}{x} \) for \( x \in (0, \infty) \)

\[
y(1) = 1, \quad \partial_x y = -y^2 \quad \sim \quad y(x) = \frac{1}{x}
\]

**Inverse function**
Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains):

- analytic
- contains polynomials, \( \sin, \cos, \tanh, \exp, \ldots \)
- stable under \( \pm, \times, /, \circ \) and Initial Value Problems (IVP)
- technicality on the field \( \mathbb{K} \) of coefficients for stability under \( \circ \)
- requires partial differential equations

Exercice: are all analytic functions generable?

Riemann \( \Gamma \) and \( \zeta \) are not generable.
Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

- analytic
- contains polynomials, sin, cos, tanh, exp, ...
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
- requires partial differential equations

Exercice : are all analytic functions generable?
Generable functions: summary

Nice theory for the class of multivariate generable functions (over connected domains):

- analytic
- contains polynomials, sin, cos, tanh, exp, ...
- stable under ±, ×, /, ◦ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under ◦
- requires partial differential equations

Exercice: are all analytic functions generable? No
Riemann $\Gamma$ and $\zeta$ are not generable.
Why is this useful?

Writing polynomial ODEs by hand is hard.
Why is this useful?

Writing polynomial ODEs by hand is hard.

Using generable functions, we can build complicated multivariate partial functions using other operations, and we know they are solutions to polynomial ODEs by construction.
Why is this useful?

Writing polynomial ODEs by hand is hard.

Using generable functions, we can build complicated multivariate partial functions using other operations, and we know they are solutions to polynomial ODEs by construction.

Example (almost rounding function)

There exists a generable function \( \text{round} \) such that for any \( n \in \mathbb{Z}, x \in \mathbb{R}, \lambda > 2 \) and \( \mu \geq 0 \):

- if \( x \in \left[ n - \frac{1}{2}, n + \frac{1}{2} \right] \) then \( |\text{round}(x, \mu, \lambda) - n| \leq \frac{1}{2} \),
- if \( x \in \left[ n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda} \right] \) then \( |\text{round}(x, \mu, \lambda) - n| \leq e^{-\mu} \).
There exists a fixed (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}^*_+)$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a unique solution $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$
A simplified proof

\[ \alpha \in \mathbb{R} \rightarrow \text{ODE} \rightarrow 011010100111 \ldots \rightarrow t \]

This is the ideal curve, the real one is an approximation of it.
A simplified proof

- \( \alpha \in \mathbb{R} \)
- \( \alpha \) ∈ \( \mathbb{R} \)
- ODE
- binary stream generator
- digits of \( \alpha \)
- \( 011010100111 \ldots \)
- \( t \)
- "Digital" to Analog Converter (fixed frequency)

Approximate Lipschitz and bounded functions with fixed precision.

NOTE: That's the trickiest part.
A simplified proof

\[ \alpha \in \mathbb{R} \xrightarrow{\text{ODE}} \text{digits of } \alpha \]

We need something more: a **fast-growing** ODE.
A simplified proof

\[ \alpha \in \mathbb{R} \rightarrow \text{ODE} \rightarrow \begin{array}{c} 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ \ldots \end{array} \rightarrow t \]

We need something more: an \textbf{arbitrarily fast-growing} ODE.

"Digital" to Analog Converter (fixed frequency)
A less simplified proof

binary stream generator: digits of $\alpha \in \mathbb{R}$

$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2} \tanh(\mu \sin(2\alpha \pi 4^{\text{round}(t-1/4, \lambda)} + 4\pi/3))$$

It's horrible, but generable

```
1 0 1 0 1 0 1 0
```

`round` is the mysterious rounding function...
A less simplified proof

binary stream generator: digits of $\alpha \in \mathbb{R}$

$\begin{array}{cccccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}$

$\frac{9}{10}$

dyadic stream generator: $d_i = m_i 2^{-d_i}, a_i = 9i + \sum_{j<i} d_j$

$f(\alpha, \gamma, t) = \sin(2\alpha \pi 2^{\text{round}(t-1/4,\gamma)})$

$\text{round}$ is the mysterious rounding function...
A less simplified proof
A less simplified proof

copy signal
A less simplified proof
A less simplified proof
A less simplified proof
A less simplified proof

This copy operation is the “non-trivial” part.
We can do \textit{almost piecewise constant functions}...
We can do almost piecewise constant functions...

- ...that are bounded by 1...
- ...and have super slow changing frequency.
A less simplified proof

We can do almost piecewise constant functions...

- ...that are bounded by 1...
- ...and have super slow changing frequency.

How do we go to arbitrarily large and growing functions? Can a polynomial ODE even have arbitrary growth?
An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[ y_1' = y_1 \quad \sim \quad y_1(t) = \exp(t) \]
Building a fast-growing ODE, that exists over $\mathbb{R}$:

\begin{align*}
  y_1' &= y_1 \quad \leadsto \quad y_1(t) = \exp(t) \\
  y_2' &= y_1 y_2 \quad \leadsto \quad y_1(t) = \exp(\exp(t))
\end{align*}
An old question on growth

Building a fast-growing ODE, that exists over \( \mathbb{R} \):

\[
\begin{align*}
y'_1 &= y_1 \\ y'_2 &= y_1 y_2 \\ &\vdots \\ y'_n &= y_1 \cdots y_n
\end{align*}
\]

\[
\begin{align*}
y_1(t) &= \exp(t) \\ y_1(t) &= \exp(\exp(t)) \\ &\vdots \\ y_n(t) &= \exp(\cdots \exp(t) \cdots) := e_n(t)
\end{align*}
\]
An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

\[
\begin{align*}
y_1' &= y_1 & \Rightarrow & & y_1(t) = \exp(t) \\
y_2' &= y_1 y_2 & \Rightarrow & & y_1(t) = \exp(\exp(t)) \\
\vdots & & \vdots & & \vdots \\
y_n' &= y_1 \cdots y_n & \Rightarrow & & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t)
\end{align*}
\]

Conjecture (Emil Borel, 1899)

With $n$ variables, cannot do better than $O_t(e_n(At^k))$. 
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

**Conjecture (Emil Borel, 1899)**

With \( n \) variables, cannot do better than \( O_t(e_n(At^k)) \).

**Counter-example (Vijayaraghavan, 1932)**

\[
\frac{1}{2 - \cos(t) - \cos(\alpha t)}
\]

Sequence of **arbitrarily growing** spikes.
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

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**Counter-example (Vijayaraghavan, 1932)**

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\]

Sequence of **arbitrarily growing** spikes. But not good enough for us.
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

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**Theorem (In the paper)**

There exists a polynomial \( p: \mathbb{R}^d \to \mathbb{R}^d \) such that for any continuous function \( f: \mathbb{R}_+ \to \mathbb{R} \), we can find \( \alpha \in \mathbb{R}^d \) such that

satisfies

\[
y(0) = \alpha, \quad y'(t) = p(y(t))
\]

\[
y_1(t) \geq f(t), \quad \forall t \geq 0.
\]

Note: both results require \( \alpha \) to be transcendental. Conjecture still open for rational (or algebraic) coefficients.
An old question on growth

\[ e_n(t) = \exp(\cdots \exp(t) \cdots) \quad (n \text{ compositions}) \]

**Conjecture (Emil Borel, 1899)**

With \( n \) variables, cannot do better than \( \mathcal{O}_t(e_n(At^k)) \).

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**Theorem (In the paper)**

There exists a polynomial \( p : \mathbb{R}^d \to \mathbb{R}^d \) such that for any continuous function \( f : \mathbb{R}_+ \to \mathbb{R} \), we can find \( \alpha \in \mathbb{R}^d \) such that

\[ y(0) = \alpha, \quad y'(t) = p(y(t)) \]

satisfies

\[ y_1(t) \geq f(t), \quad \forall t \geq 0. \]

**Note**: both results require \( \alpha \) to be transcendental. Conjecture still open for rational (or algebraic) coefficients.
Assume $f$ is generable, can we iterate $f$ with an ODE?
That is, build a generable $y$ such that $y(x, n) \approx f^n(x)$ for all $n \in \mathbb{N}$.
Assume $f$ is generable, can we iterate $f$ with an ODE? That is, build a generable $y$ such that $y(x, n) \approx f[n](x)$ for all $n \in \mathbb{N}$.
Assume $f$ is generable, can we iterate $f$ with an ODE? That is, build a generable $y$ such that $y(x, n) \approx f^n(x)$ for all $n \in \mathbb{N}$.
Assume $f$ is generable, can we iterate $f$ with an ODE? That is, build a generable $y$ such that $y(x, n) \approx f^n(x)$ for all $n \in \mathbb{N}$.
There exists a fixed (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}^*_+)$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a unique solution $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R},$

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$

Furthermore, $\alpha$ is computable from $f$ and $\varepsilon$.  

Remarks:

- if $f$ and $\varepsilon$ are computable then $\alpha$ is computable
- if $f$ or $\varepsilon$ is not computable then $\alpha$ is not computable
- in all cases $\alpha$ is a horrible transcendental number