A universal differential equation

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Joint work with Olivier Bournez and Daniel Graça

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INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

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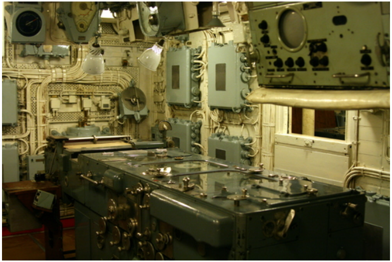
What is a computer?

What is a computer ?

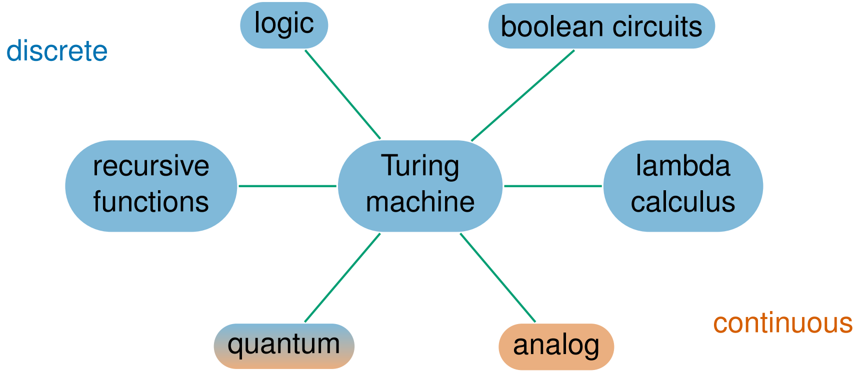
What is a computer ?

VS



Church Thesis

Computability

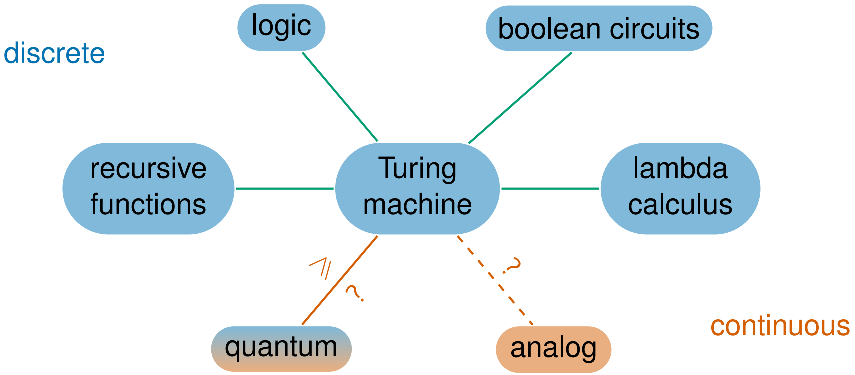


Church Thesis

All **reasonable** models of computation are equivalent.

Church Thesis

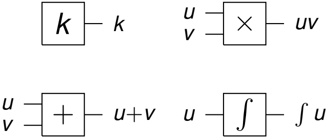
Complexity



Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.

Polynomial Differential Equations



General Purpose
Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks :

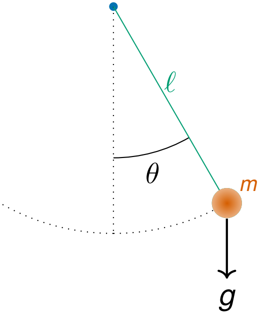
- chemical
- enzymatic

polynomial differential equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable ($+,\times,\circ,/$,ED)
- No closed-form solution

Example of dynamical system



$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

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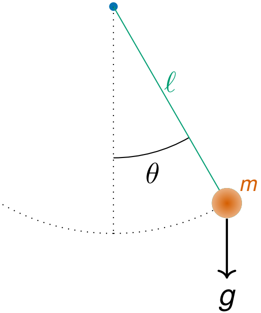

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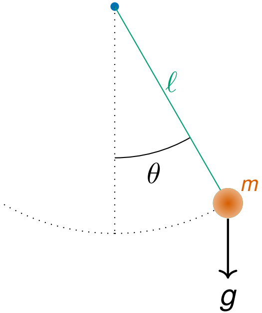

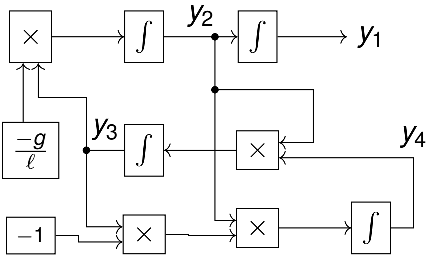

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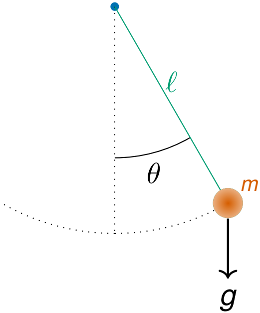

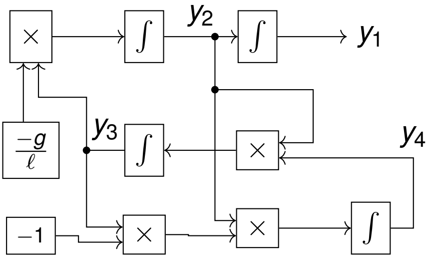

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Historical remark : the word “analog”

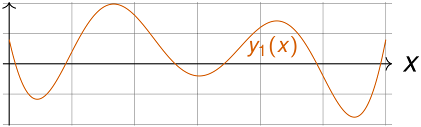
The pendulum and the circuit have the same equation. One can study one using the other by analogy.

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$f(x) = y_1(x)$



Shannon's notion

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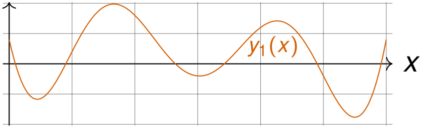

Shannon's notion

$\sin, \cos, \exp, \log, \ldots$

Strictly weaker than Turing machines [Shannon, 1941]

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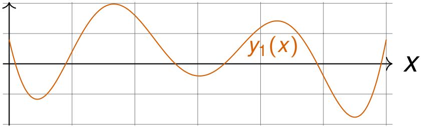

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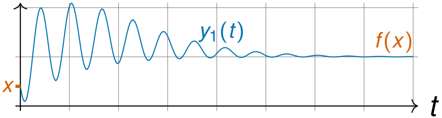
Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$



Modern notion

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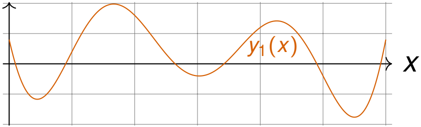

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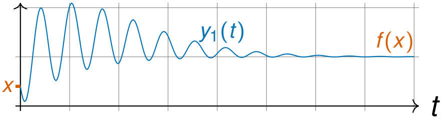


Modern notion

sin, cos, exp, log, Γ, ζ, ...

Turing powerful [Bournez et al., 2007]

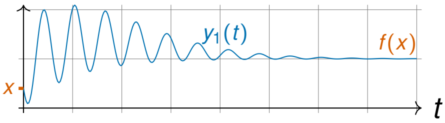
Equivalence with computable analysis

Definition (Bournez et al, 2007)

f **computable by GPAC** if $\exists p$ polynomial such that $\forall x \in [a, b]$

$$y(0) = (x, 0, \dots, 0) \qquad y'(t) = p(y(t))$$

satisfies $|f(x) - y_1(t)| \leqslant y_2(t)$ et $y_2(t) \xrightarrow[t\to\infty]{} 0$.



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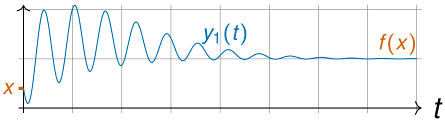

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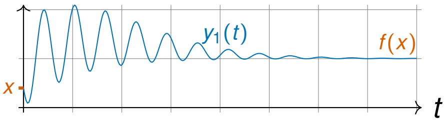

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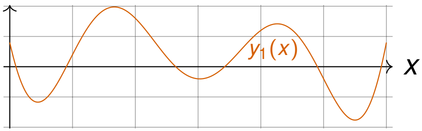
1. In Computable Analysis, a standard model over reals built from Turing machines.

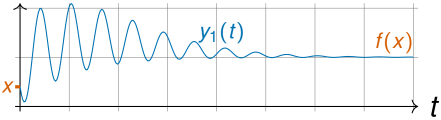
Universal differential equations

Generable functions



subclass of analytic functions

Computable functions



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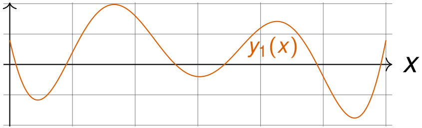

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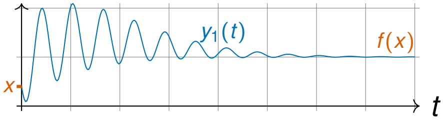

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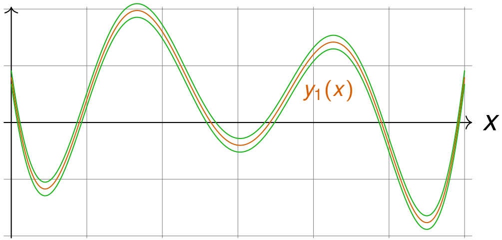
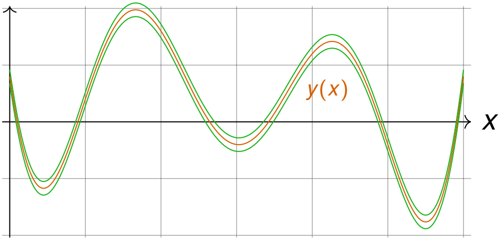


Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

For any continuous functions f and ε, there exists $y : \mathbb{R} \to \mathbb{R}$ solution to

$$3y'^4y''y''''^2 - 4y'^4y'''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y''^3y'''^2 + 12y''^7 = 0$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

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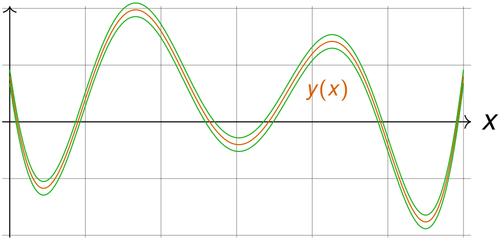

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*There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to*

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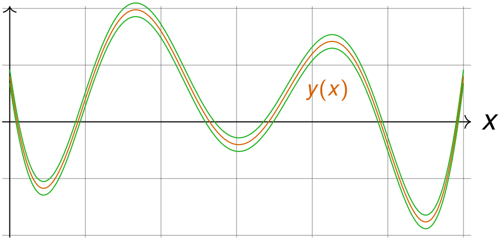

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Problem : this is «weak» result.

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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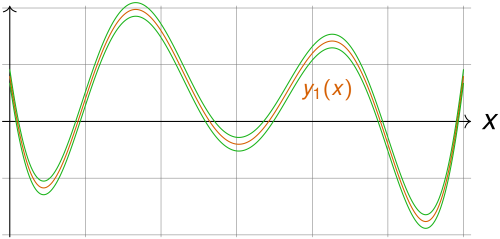
- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Universal initial value problem (IVP)



Theorem

*There exists a **fixed** (vector of) polynomial p such that for any continuous functions f and ε, there exists $\alpha \in \mathbb{R}^d$ such that*

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

*has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,*

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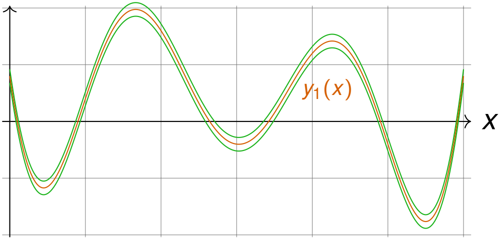

Notes :

- **system** of ODEs,
- y is analytic,
- we need $d \approx 300$.

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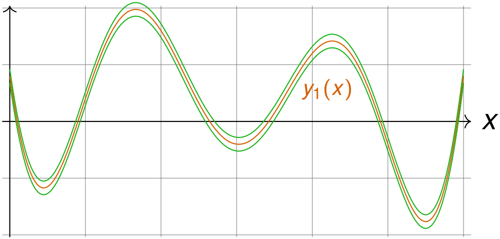

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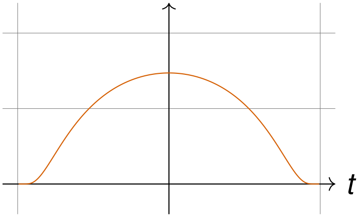
Remark : α is usually transcendental, but computable from f and ε

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

$$\text{It satisfies} \quad (1-t^2)^2 f^{''}(t) + 2tf'(t) = 0.$$



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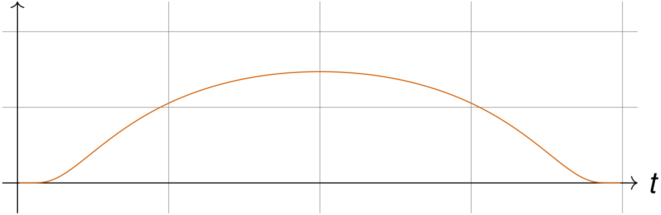
- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at+b)$ satisfies

$$\begin{aligned} 3y'^4 y'' y''''^2 \quad & -4y'^4 y''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ & -12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 = 0 \end{aligned}$$

Translation and rescaling :



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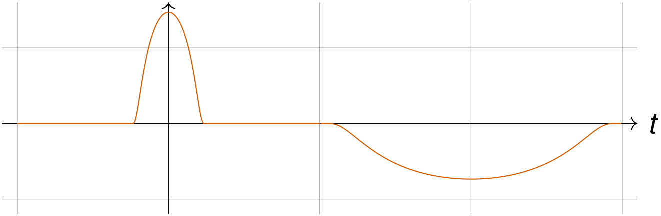
- Can glue together arbitrary many such pieces



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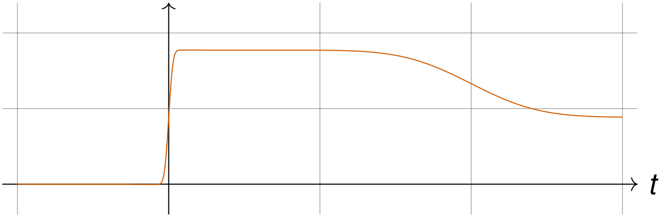
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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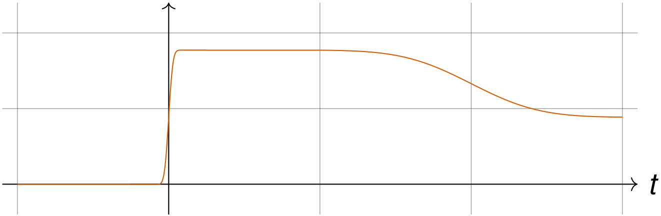
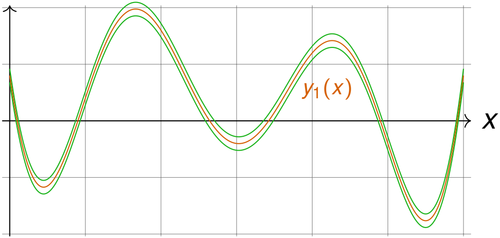


Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

Universal DAE revisited



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$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

*has a **unique analytic solution** and this solution satisfies such that*

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

A brief stop

Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by programming with ODEs.

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \qquad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

Types

- $d \in \mathbb{N}$: dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
- $y_0 \in \mathbb{K}^d, y : \mathbb{R} \to \mathbb{R}^d$

Note : existence and unicity of y by Cauchy-Lipschitz theorem.

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists d, p and y_0 such that the solution y to

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Example : $f(x) = x$ ▶ identity

$$y(0) = 0, \qquad y' = 1 \quad \rightsquigarrow \quad y(x) = x$$

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Example : $f(x) = x^2$ ▶ squaring

$$\begin{aligned} y_1(0) &= 0, & y_1' &= 2y_2 & \rightsquigarrow \quad y_1(x) &= x^2 \\ y_2(0) &= 0, & y_2' &= 1 & \rightsquigarrow \quad y_2(x) &= x \end{aligned}$$

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Example : $f(x)=x^n$ ▶ n^{th} power

$$\begin{array}{lll} y_1(0)=0, & y_1'=ny_2 & \rightsquigarrow\ y_1(x)=x^n \\ y_2(0)=0, & y_2'=(n-1)y_3 & \rightsquigarrow\ y_2(x)=x^{n-1} \\ \dots & \dots & \dots \\ y_n(0)=0, & y_n=1 & \rightsquigarrow\ y_n(x)=x \end{array}$$

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Example : $f(x)=\exp(x)$ ▶ exponential

$$y(0)=1,\qquad y'=y\quad\rightsquigarrow\quad y(x)=\exp(x)$$

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Example : $f(x)=\sin(x)$ or $f(x)=\cos(x)$

- sine/cosine

$$\begin{array}{llll} y_1(0)=0, & y_1'=y_2 & \rightsquigarrow & y_1(x)=\sin(x) \\ y_2(0)=1, & y_2'=-y_1 & \rightsquigarrow & y_2(x)=\cos(x) \end{array}$$

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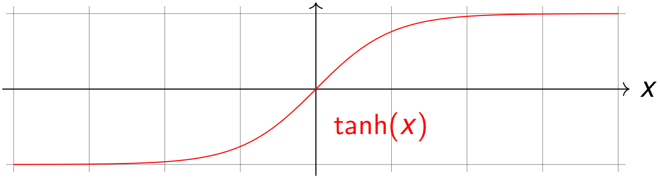
Example : $f(x) = \tanh(x)$ ▸ hyperbolic tangent

$$y(0) = 0, \qquad y' = 1 - y^2 \quad \rightsquigarrow \quad y(x) = \tanh(x)$$



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Example : $f(x)=\frac{1}{1+x^2}$ ▶ rational function

$$f'(x)=\frac{-2x}{(1+x^2)^2}=-2xf(x)^2$$

$$\begin{array}{lll} y_1(0)=1, & y_1'=-2y_2y_1^2 & \rightsquigarrow\quad y_1(x)=\frac{1}{1+x^2} \\ y_2(0)=0, & y_2'=1 & \rightsquigarrow\quad y_2(x)=x \end{array}$$

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Example : $f=g\pm h$ ▶ sum/difference

$$(g\pm h)'=g'\pm h'$$

assume :

$$z(0)=z_0,\qquad z'=p(z)\qquad\rightsquigarrow\qquad z_1=g$$
$$w(0)=w_0,\qquad w'=q(w)\qquad\rightsquigarrow\qquad w_1=h$$

then :

$$y(0)=z_{0,1}+w_{0,1},\qquad y'=p_1(z)\pm q_1(w)\qquad\rightsquigarrow\qquad y=z_1\pm w_1$$

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Example : $f = gh$ ▶ product

$$(gh)' = g'h + gh'$$

assume :

$$z(0) = z_0, \qquad z' = p(z) \qquad \rightsquigarrow \qquad z_1 = g$$
$$w(0) = w_0, \qquad w' = q(w) \qquad \rightsquigarrow \qquad w_1 = h$$

then :

$$y(0) = z_{0,1} w_{0,1}, \qquad y' = p_1(z) w_1 + z_1 q_1(w) \qquad \rightsquigarrow \qquad y = z_1 w_1$$

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Example : $f = \frac{1}{g}$ ▶ inverse

$$f' = \frac{-g'}{g^2} = -g' f^2$$

assume :

$$z(0) = z_0, \qquad z' = p(z) \qquad \rightsquigarrow \qquad z_1 = g$$

then :

$$y(0) = \frac{1}{z_{0,1}}, \qquad y' = -p_1(z) y^2 \qquad \rightsquigarrow \qquad y = \frac{1}{z_1}$$

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Example : $f=\int g$ ▶ integral

assume :

$$z(0)=z_0,\qquad z'=p(z)\quad\rightsquigarrow\quad z_1=g$$

then :

$$y(0)=0,\qquad y'=z_1\quad\rightsquigarrow\quad y=\int z_1$$

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Example : $f = g'$ ▶ derivative

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

assume :

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$$y(0) = p_1(z_0), \qquad y' = \nabla p_1(z) \cdot p(z) \quad \rightsquigarrow \quad y = z_1''$$

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Example : $f = g \circ h$ ▶ composition

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

assume :

$$\begin{array}{lll} z(0) = z_0, & z' = p(z) & \rightsquigarrow \quad z_1 = g \\ w(0) = w_0, & w' = q(w) & \rightsquigarrow \quad w_1 = h \end{array}$$

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$$y(0) = z(w_0), \qquad y' = p(y)z_1 \quad \rightsquigarrow \quad y = z \circ h$$

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Is this coefficient in $\mathbb{K}$?

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then :

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Is this coefficient in $\mathbb{K}$? Fields with this property are called generable.

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Example : $f'=\tanh\circ f$ ▶ Non-polynomial differential equation

$$f''=(\tanh'\circ f)f'=(1-(\tanh\circ f)^2)f'$$

$$\begin{array}{lll} y_1(0)=f(0), & y_1'=y_2 & \rightsquigarrow\quad y_1(x)=f(x)\\ y_2(0)=\tanh(f(0)), & y_2'=(1-y_2^2)y_2 & \rightsquigarrow\quad y_2(x)=\tanh(f(x))\end{array}$$

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Example : $f(0) = f_0, f' = g \circ f$ ▶ Initial Value Problem (IVP)

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

assume :

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$$y(0) = p_1(z_0), \qquad y' = \nabla p_1(z) \cdot p(z) \quad \rightsquigarrow \quad y = z_1''$$

Generable functions : a first summary

Nice theory for the class of total and univariate generable functions :

- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
- solutions to polynomial ODEs form a **very large class**

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Limitations :

- total functions
- univariate

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \qquad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

$J_y(x) =$ Jacobian matrix of y at x

Types

- $n \in \mathbb{N}$: input dimension
- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{K}^{d \times d}[\mathbb{R}^d]$: polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d, y : X \to \mathbb{R}^d$

Notes :

- Partial differential equation !
- Unicity of solution y...
- ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3)$ ▸ monomial

$$y(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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Example : $f(x_1, x_2) = x_1 x_2^2$ ► monomial

$$\begin{array}{llll} y_1(0,0) = 0, & \partial_{x_1} y_1 = y_3^2, & \partial_{x_2} y_1 = 3 y_2 y_3 & \rightsquigarrow \quad y_1(x) = x_1 x_2^2 \\ y_2(0,0) = 0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \rightsquigarrow \quad y_2(x) = x_1 \\ y_3(0,0) = 0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \rightsquigarrow \quad y_3(x) = x_2 \end{array}$$

This is tedious !

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Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$ ▶ inverse function

$$y(1) = 1, \qquad \partial_x y = -y^2 \quad \rightsquigarrow \quad y(x) = \frac{1}{x}$$

Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$, ...
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
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Exercice : are all analytic functions generable ?

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Exercice : are all analytic functions generable ? No
Riemann Γ and ζ are not generable.

Why is this useful?

Writing polynomial ODEs by hand is hard.

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Using generable functions, we can build complicated **multivariate partial functions** using other operations, and we know they are solutions to polynomial ODEs **by construction.**

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Using generable functions, we can build complicated **multivariate partial functions** using other operations, and we know they are solutions to polynomial ODEs **by construction.**

Example (almost rounding function)

There exists a generable function round such that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $\lambda > 2$ and $\mu \geqslant 0$:

- if $x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant \frac{1}{2}$,
- if $x \in \left[n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant e^{-\mu}$.

Reminder of the result

Main result (reminder)

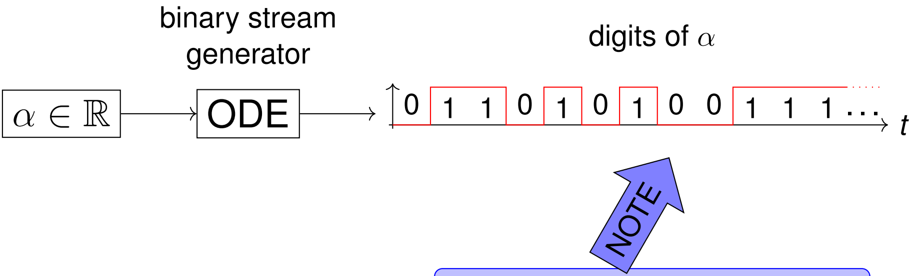
There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_+^*)$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

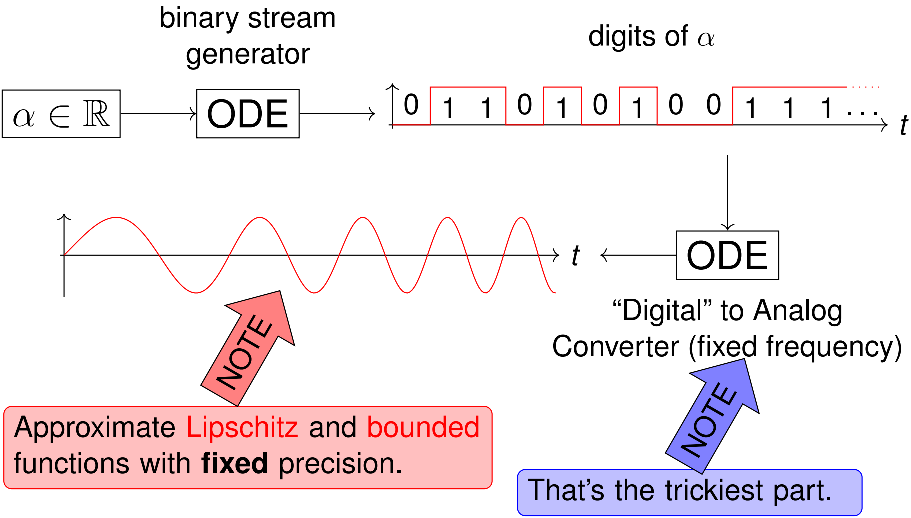
$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

A simplified proof

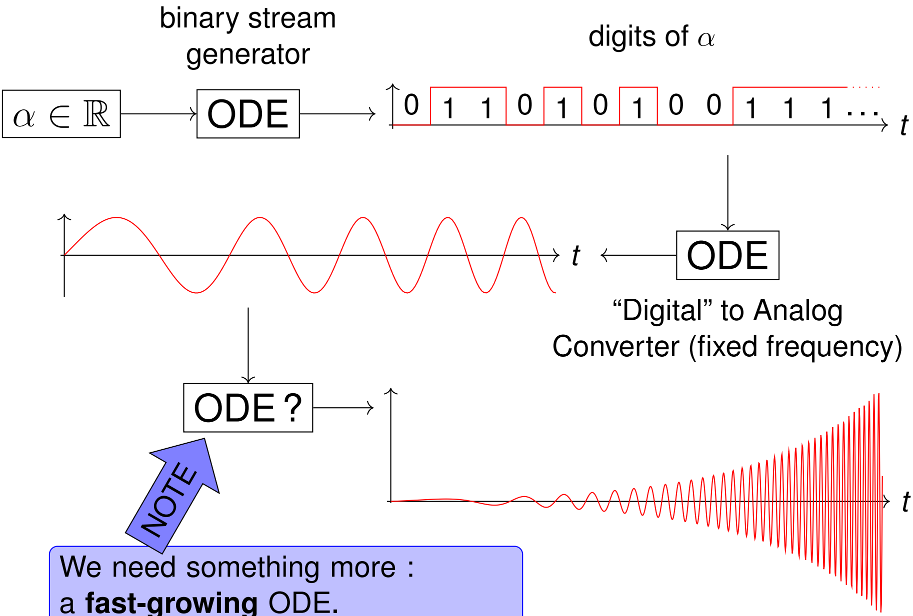


This is the ideal curve, the real one is an approximation of it.

A simplified proof

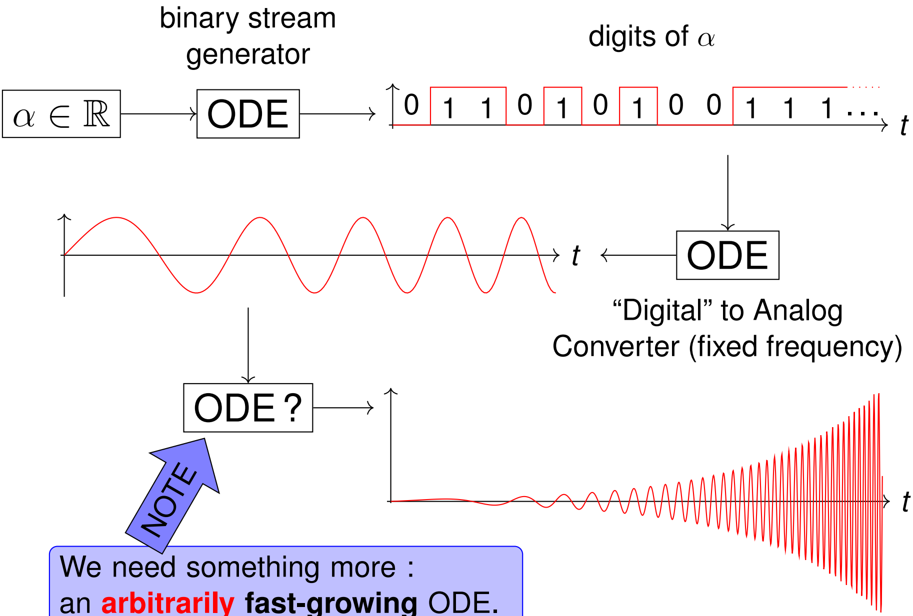


A simplified proof



We need something more :
a **fast-growing** ODE.

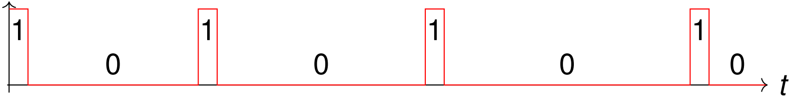
A simplified proof



We need something more :
an **arbitrarily** **fast-growing** ODE.

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



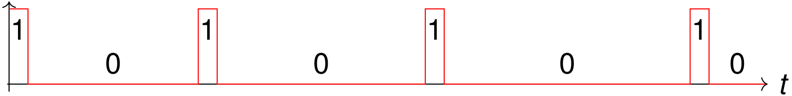
$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2}\tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4,\lambda)} + 4\pi/3))$$

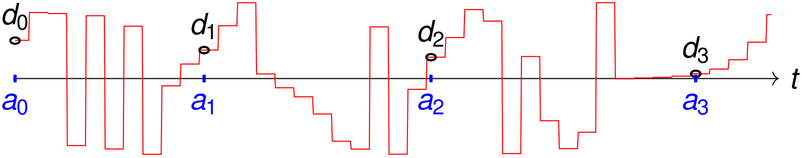
It's horrible, but generable

round is the mysterious rounding function...

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



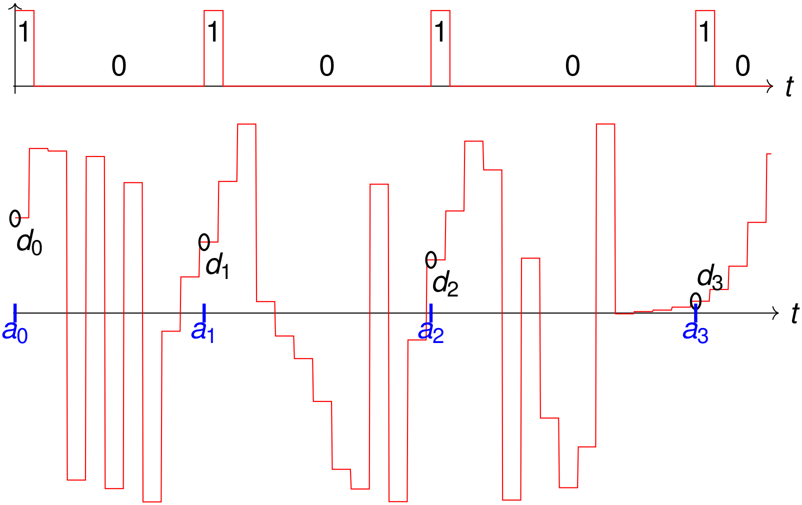


dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j<i} d_j$

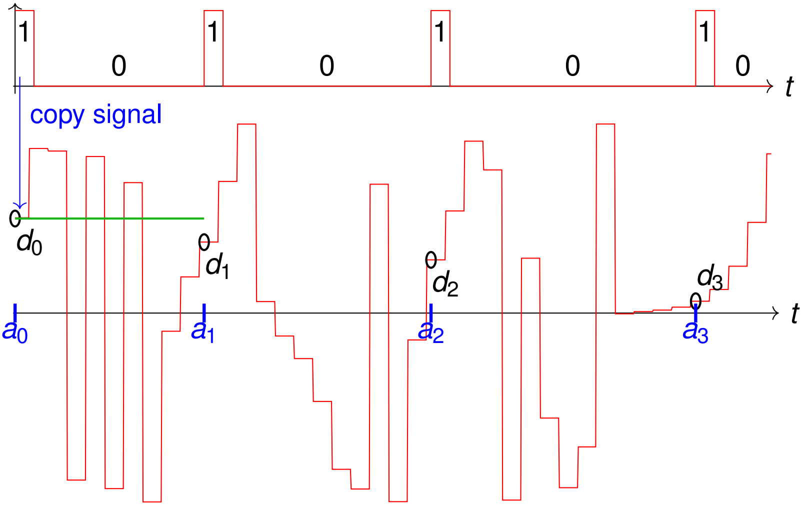
$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4,\gamma)}))$$

round is the mysterious rounding function...

A less simplified proof



A less simplified proof



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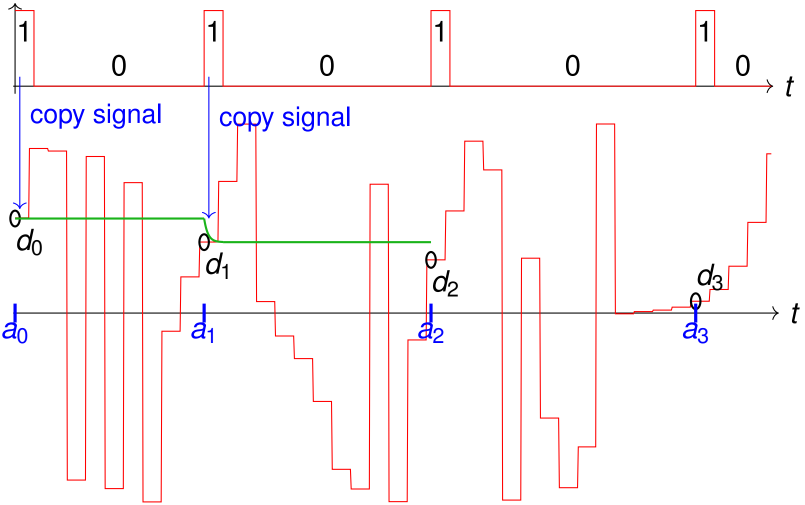

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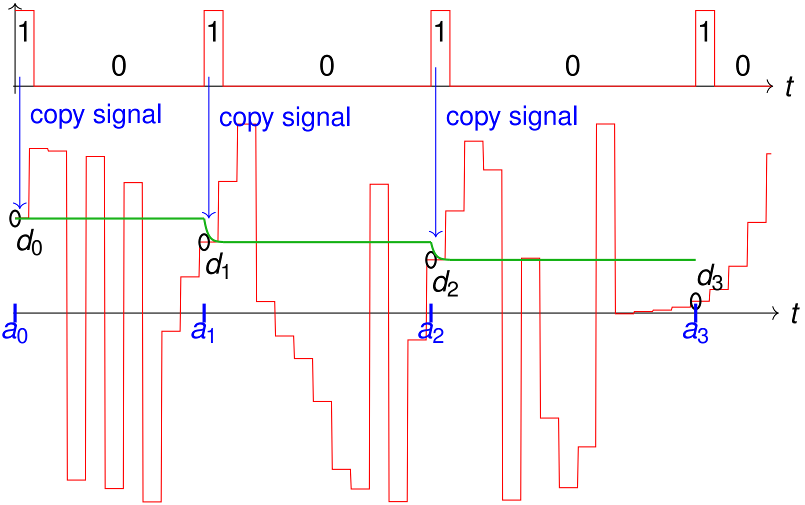

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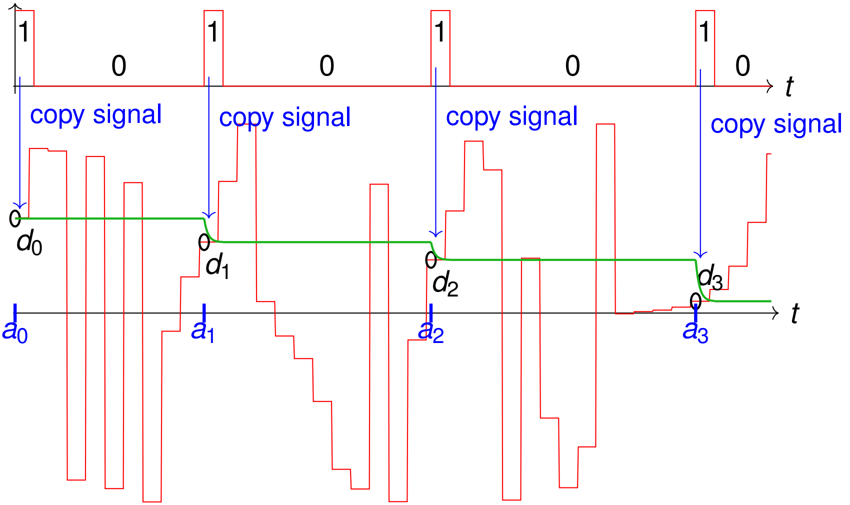

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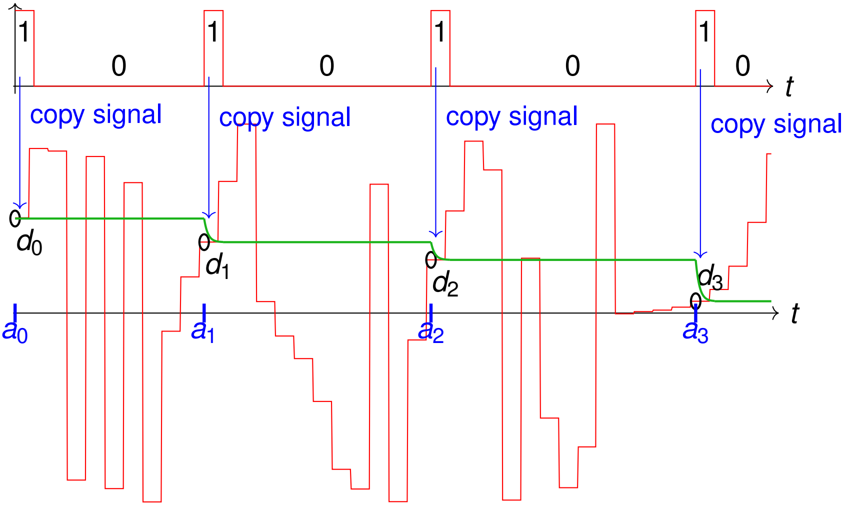
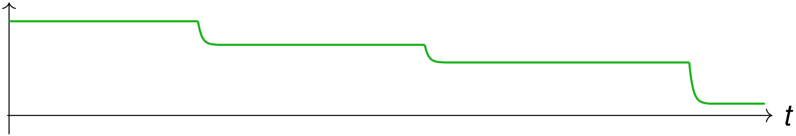


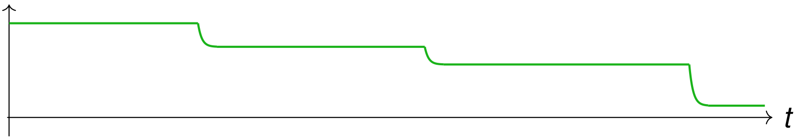
This copy operation is the “non-trivial” part.

A less simplified proof



We can do **almost piecewise constant functions...**

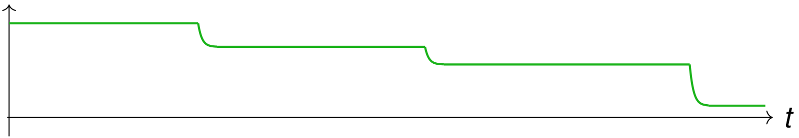
A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

How do we go to arbitrarily large and growing functions ? Can a polynomial ODE even have arbitrary growth ?

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \qquad \rightsquigarrow \qquad y_1(t) = \exp(t)$$

An old question on growth

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$$\begin{array}{lcl} y_1' = y_1 & \rightsquigarrow & y_1(t) = \exp(t) \\ y_2' = y_1 y_2 & \rightsquigarrow & y_1(t) = \exp(\exp(t)) \end{array}$$

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Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

An old question on growth

$e_n(t) = \exp(\cdots\exp(t)\cdots)$ (n compositions)

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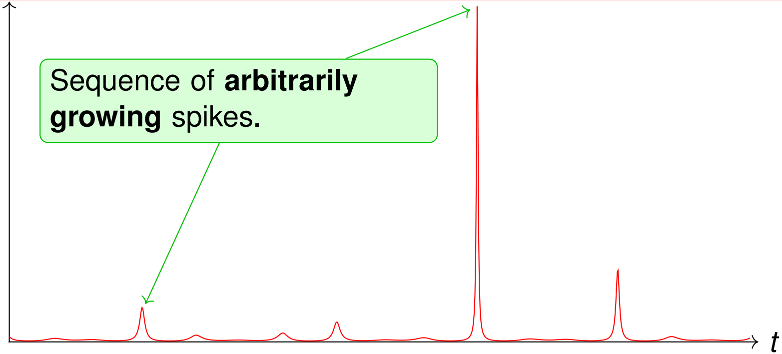
Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2-\cos(t)-\cos(\alpha t)}$$



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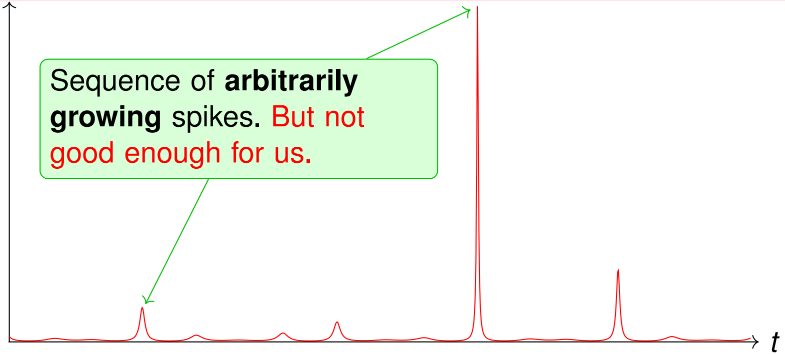

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Theorem (In the paper)

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_+ \to \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

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Note : both results require α to be **transcendental**. Conjecture still open for **rational** (or algebraic) coefficients.

Proof gem : iteration with differential equations

Assume f is generable, can we **iterate** f with an ODE ?
That is, build a generable y such that $y(x,n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$

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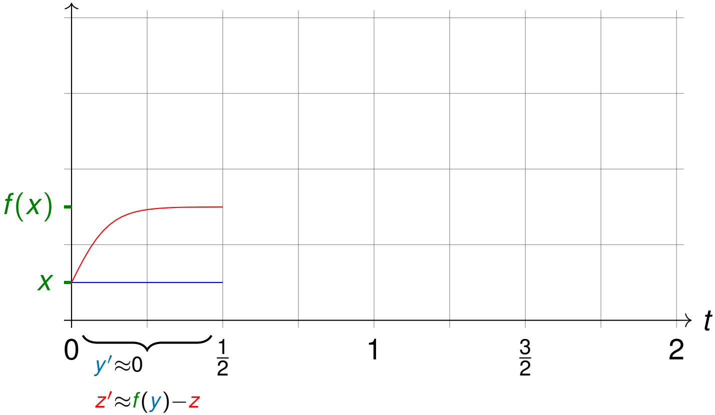

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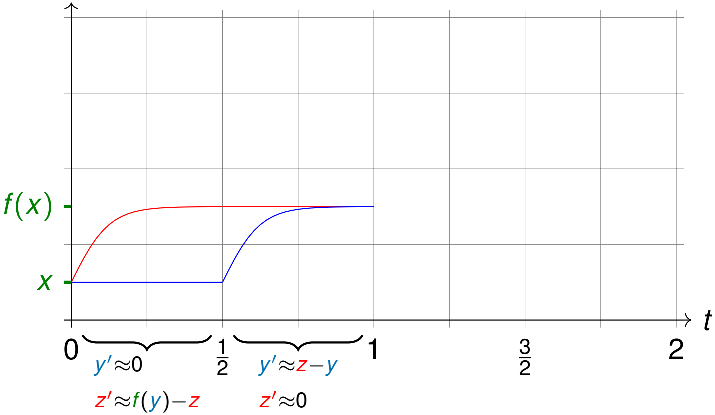

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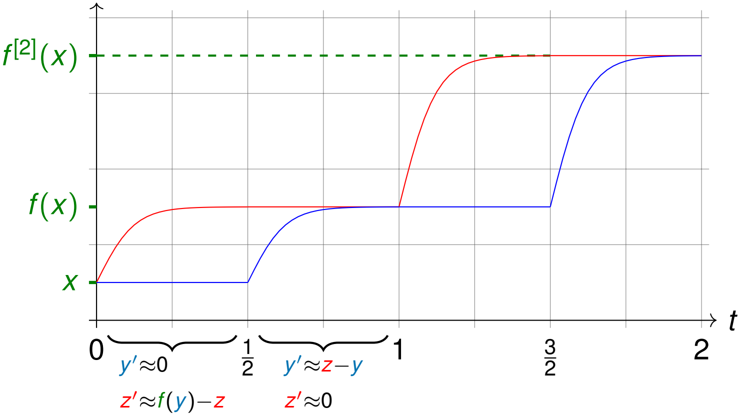

Main result, remark and end

Main result (reminder)

There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}^*_+)$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Futhermore, α is computable from f and ε.

Remarks :

- if f and ε are computable then α is computable
- if f or ε is **not computable** then α is **not computable**
- in all cases α is a horrible transcendental number