Continuous models of computation: computability, complexity, universality

Amaury Pouly

Université de Paris, IRIF, CNRS

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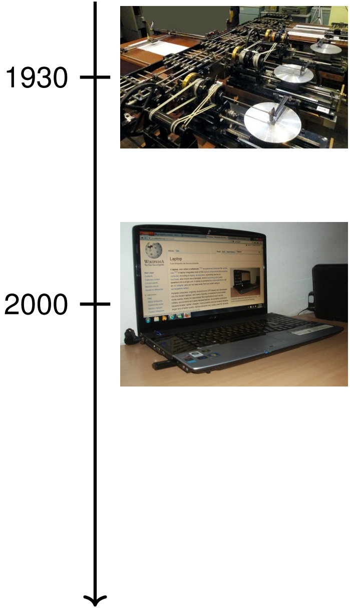
Analog computers : the come back !

1930

analog computers :

- hard to program
- highly specialized

Analog computers : the come back !



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digital computers :

- « easy » to program
- general purpose

Analog computers : the come back !

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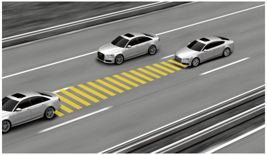

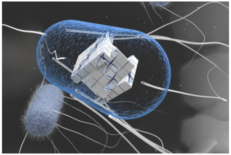

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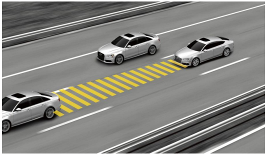
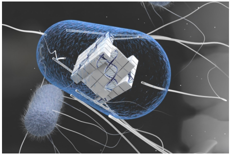
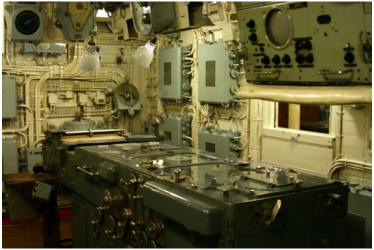




- analog ?
- digital ?
- both!

Analog computers

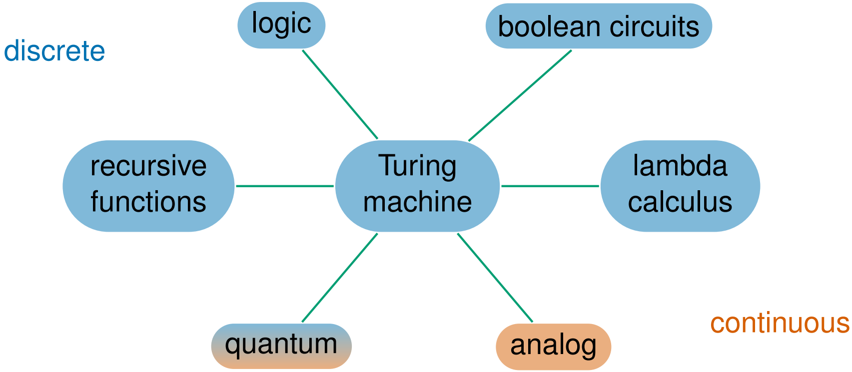
Differential Analyser
“Mathematica of 1920”



Admiralty Fire Control Table
British Navy (WW2)

Church Thesis

Computability

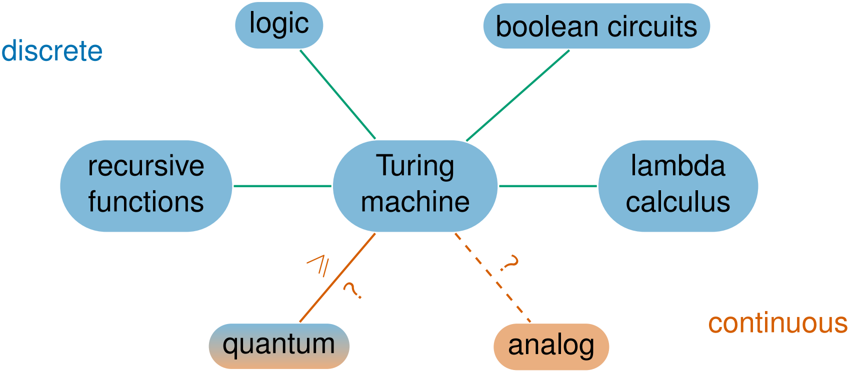


Church Thesis

All **reasonable** models of computation are equivalent.

Church Thesis

Complexity



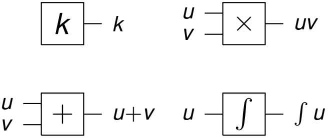
Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.

From machines to models

Differential analyzer

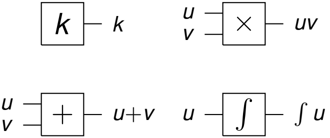
From machines to models



General Purpose Analog Computer, Shannon 1936

Differential analyzer

From machines to models

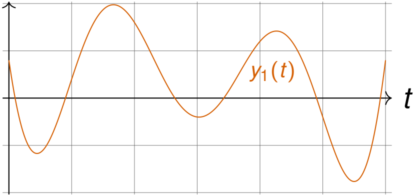


General Purpose Analog Computer, Shannon 1936

Differential analyzer

$$y(0) = y_0, \quad y'(t) = p(y(t))$$

Polynomial Differential Equation, Graça 2004



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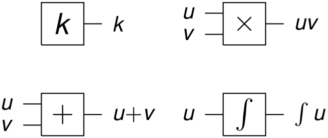

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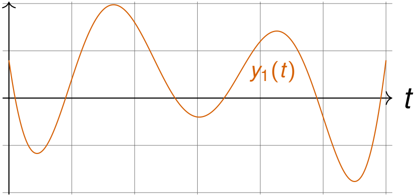
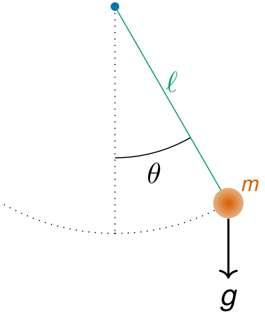


Example of dynamical system



$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

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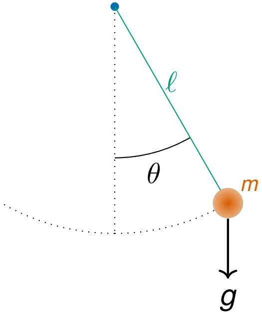

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{l} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

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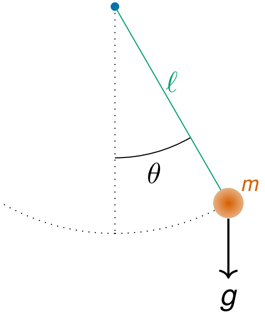

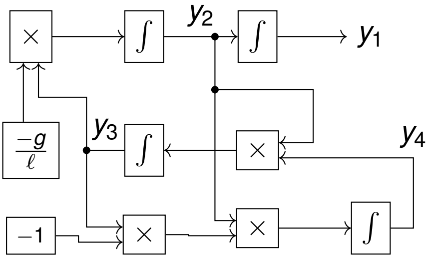

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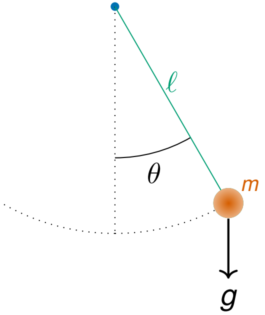

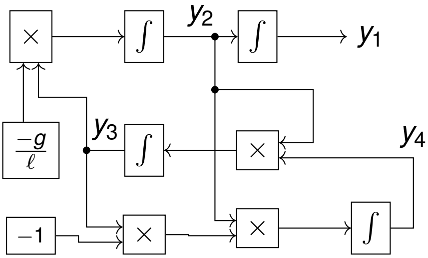

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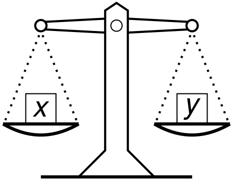
Historical remark : the word “analog”

The pendulum and the circuit have the same equation. One can study one using the other by analogy.

Does a balance scale compute a function ?

Inputs : $x, y \in [0, +\infty)$



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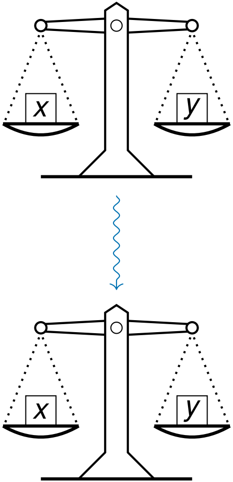

$x = y$

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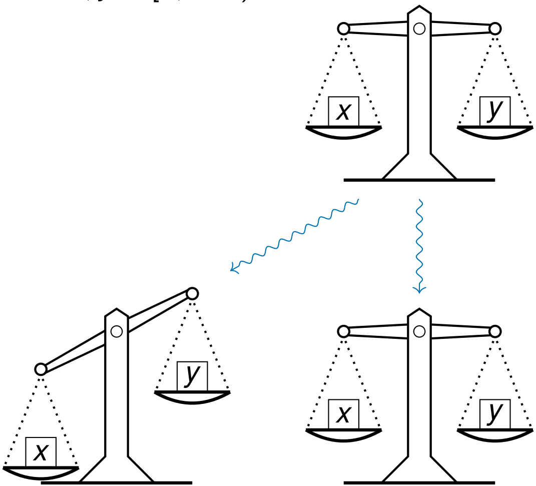

$x > y$ $x = y$

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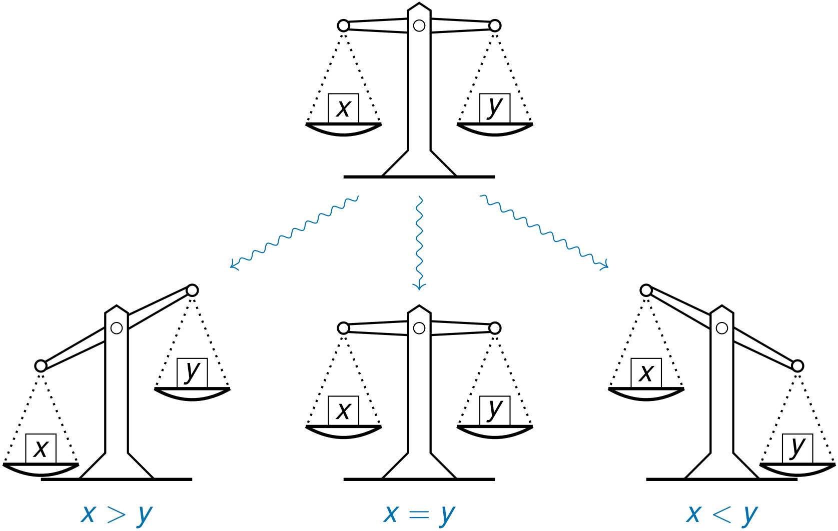

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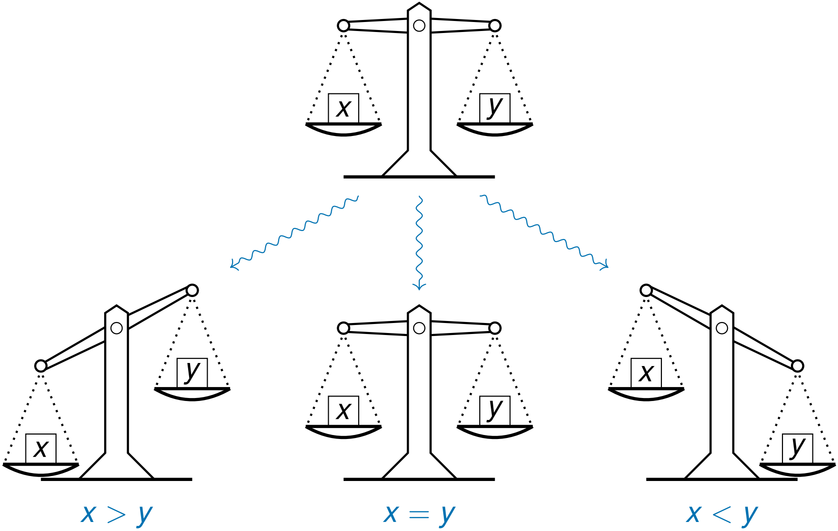


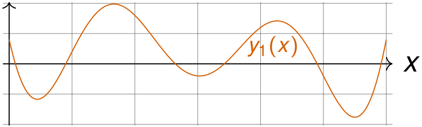
Output : $\mathrm{sign}(x - y)$?

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



Shannon's notion

sin, cos, exp, log, ...

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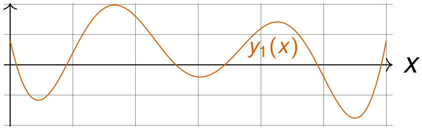

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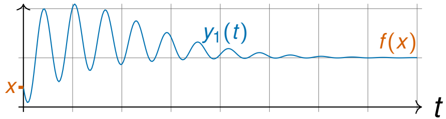
Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$



Modern notion

sin, cos, exp, log, Γ, ζ, ...

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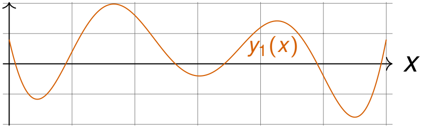

Shannon's notion

$\sin, \cos, \exp, \log, \ldots$

Considered "weak" : not Γ and ζ
Only analytic functions

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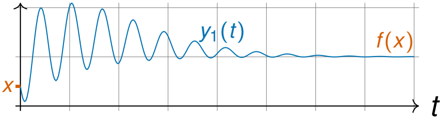

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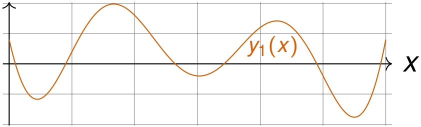

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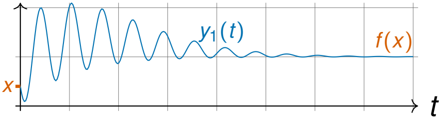
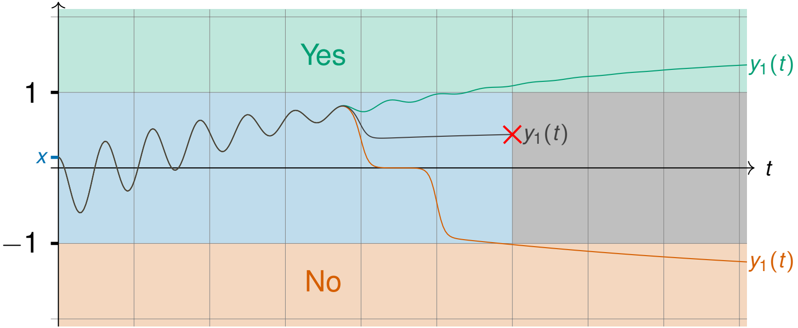


Modern notion

sin, cos, exp, log, Γ, ζ, ...

Turing powerful
[Bournez et al., 2007]

More formally



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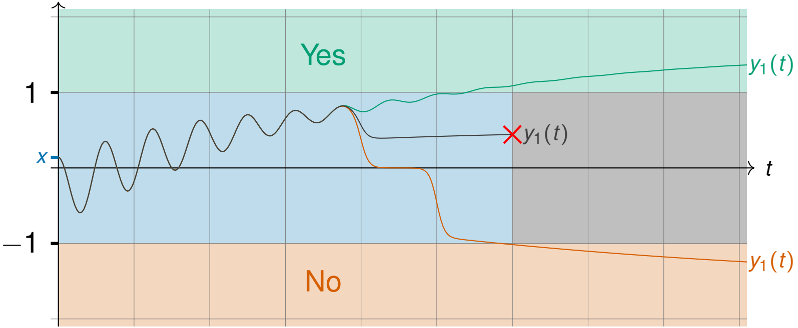
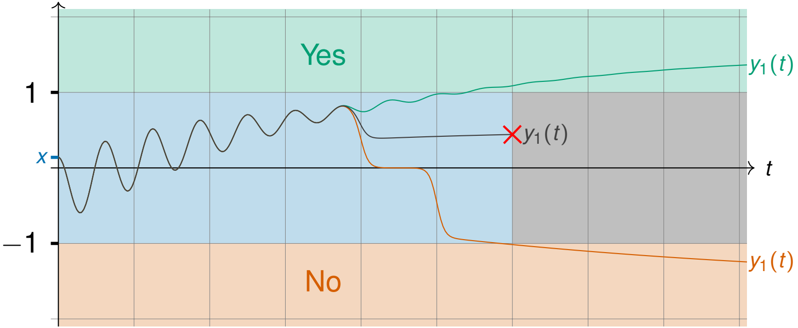


Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

More formally



Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

- analog computability theory
- purely continuous characterization of classical computability

How does one prove such a result ?

By computing/programming with differential equations ! Two levels :

Generable functions :

- « simple » basic blocks
- lots of way to combine them
- very low level

Computable functions :

- more comprehensible
- harder to combine
- higher level

The theory of generable functions

Generable functions (total, univariate)

Definition

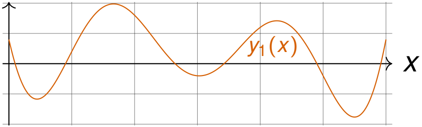
$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \qquad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

Types

- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{R}^d[\mathbb{R}^n]$: polynomial vector
- $y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d$



Note : existence and unicity of y by Cauchy-Lipschitz theorem.

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Example : $f(x) = x$ ▶ identity

$$y(0) = 0, \qquad y' = 1 \quad \rightsquigarrow \quad y(x) = x$$

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Example : $f(x) = x^2$ ► squaring

$$\begin{array}{lll} y_1(0) = 0, & y_1' = 2y_2 & \leadsto \quad y_1(x) = x^2 \\ y_2(0) = 0, & y_2' = 1 & \leadsto \quad y_2(x) = x \end{array}$$

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Example : $f(x) = x^n$ ► n^{th} power

$$\begin{array}{lll}
y_1(0) = 0, & y_1' = ny_2 & \rightsquigarrow \quad y_1(x) = x^n \\
y_2(0) = 0, & y_2' = (n-1)y_3 & \rightsquigarrow \quad y_2(x) = x^{n-1} \\
\ldots & \ldots & \qquad \ldots \\
y_n(0) = 0, & y_n = 1 & \rightsquigarrow \quad y_n(x) = x
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Example : $f(x)=\exp(x)$ ► exponential

$$y(0)=1,\qquad y'=y\quad\rightsquigarrow\quad y(x)=\exp(x)$$

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Example : $f(x)=\sin(x)$ or $f(x)=\cos(x)$

- sine/cosine

$$\begin{aligned} y_1(0)&=0, & y_1'&=y_2 & \rightsquigarrow\quad y_1(x)&=\sin(x)\\ y_2(0)&=1, & y_2'&=-y_1 & \rightsquigarrow\quad y_2(x)&=\cos(x)\end{aligned}$$

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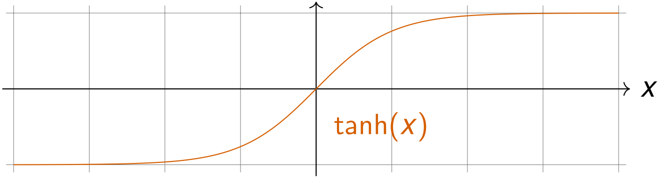
Example : $f(x) = \tanh(x)$ ► hyperbolic tangent

$$y(0) = 0, \qquad y' = 1 - y^2 \quad \rightsquigarrow \quad y(x) = \tanh(x)$$



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Example : $f(x)=\frac{1}{1+x^2}$ ▶ rational function

$$f'(x)=\frac{-2x}{(1+x^2)^2}=-2xf(x)^2$$

$$\begin{aligned} y_1(0)&=1, & y_1'&=-2y_2y_1^2 & \rightsquigarrow\quad y_1(x)&=\frac{1}{1+x^2}\\ y_2(0)&=0, & y_2'&=1 & \rightsquigarrow\quad y_2(x)&=x \end{aligned}$$

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Example : $f=g\pm h$

- sum/difference

$$(f\pm g)'=f'\pm g'$$

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- $p\in\mathbb{R}^d[\mathbb{R}^n]$: polynomial vector
- $y_0\in\mathbb{R}^d, y:\mathbb{R}\to\mathbb{R}^d$

Example : $f=gh$ ▸ product

$$(gh)'=g'h+gh'$$

Generable functions (total, univariate)

Definition

$f:\mathbb{R}\to\mathbb{R}$ is generable if there exists d,p and y_0 such that the solution y to

$$y(0)=y_0,\qquad y'(x)=p(y(x))$$

satisfies $f(x)=y_1(x)$ for all $x\in\mathbb{R}$.

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Example : $f=\frac{1}{g}$ ► inverse

$$f'=\frac{-g'}{g^2}=-g'f^2$$

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Example : $f = \int g$ ▶ integral

$$f' = g$$

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Example : $f = g'$ ▶ derivative

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

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Example : $f = g \circ h$ ▶ composition

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

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Example : $f'=\tanh\circ f$ ► Non-polynomial differential equation

$$f''=(\tanh'\circ f)f'=(1-(\tanh\circ f)^2)f'$$

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Example : $f(0) = f_0, f' = g \circ f$ ▶ Initial Value Problem (IVP)

$$f' = g'' = (p(z))' = \nabla p(z) \cdot z'$$

Generable functions : a first summary

Nice theory for the class of total and univariate generable functions :

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$
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Limitations :

- total functions
- univariate

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \qquad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

$J_y(x) =$ Jacobian matrix of y at x

Types

- $n \in \mathbb{N}$: input dimension
- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{K}^{d\times d}[\mathbb{R}^d]$: polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d, y : X \to \mathbb{R}^d$

Notes :

- Partial differential equation !
- Unicity of solution y...
- ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3)$ ▶ monomial

$$y(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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Example : $f(x_1, x_2) = x_1 x_2^2$ ► monomial

$$\begin{array}{llll} y_1(0,0) = 0, & \partial_{x_1} y_1 = y_3^2, & \partial_{x_2} y_1 = 3y_2 y_3 & \rightsquigarrow \quad y_1(x) = x_1 x_2^2 \\ y_2(0,0) = 0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \rightsquigarrow \quad y_2(x) = x_1 \\ y_3(0,0) = 0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \rightsquigarrow \quad y_3(x) = x_2 \end{array}$$

This is tedious !

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Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$ ► inverse function

$$y(1) = 1, \qquad \partial_x y = -y^2 \quad \rightsquigarrow \quad y(x) = \frac{1}{x}$$

Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

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- analytic $\rightarrow$ isn't that very limited ?
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Riemann Γ and ζ are not generable.

Why is this useful?

Writing polynomial ODEs by hand is hard.

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Example : almost rounding function

There exists a generable function round such that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $\lambda > 2$ and $\mu \geqslant 0$:

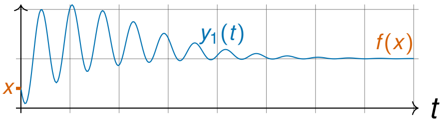
- if $x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant \frac{1}{2}$,
- if $x \in \left[n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant e^{-\mu}$.

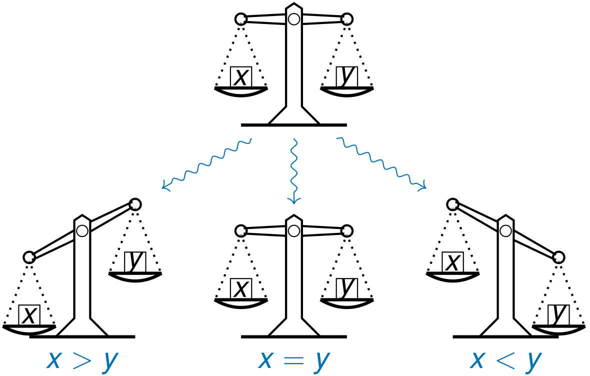
The theory of computable functions

Computable function

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t \to \infty} y_1(t)$$





Inputs : $x, y \in [0, +\infty)$ Output : $\mathrm{sign}(x - y)$?

The theory of computable functions

Important fact :

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- continuous functions

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Enough to simulate a Turing machine !

The theory of computable functions

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Enough to simulate a Turing machine !

Proof are too complicated but essentially this is all error management.

Proof gem : iteration with differential equations

Assume f is generable, can we **iterate** f with an ODE ?
That is, build a generable y such that $y(x, n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$

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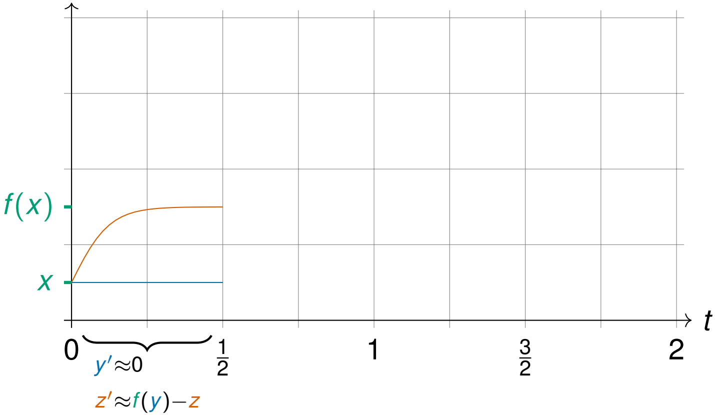

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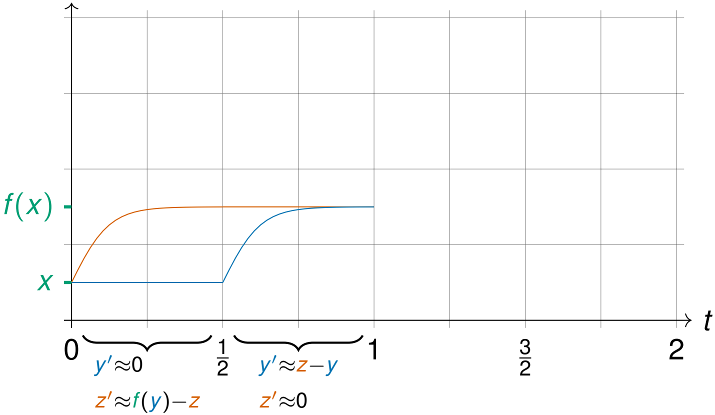

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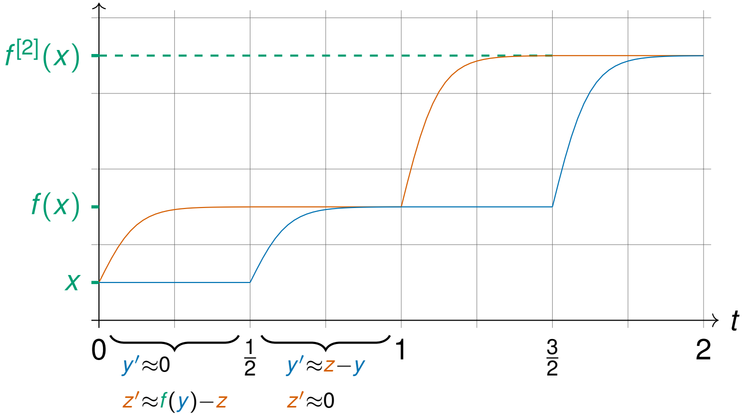
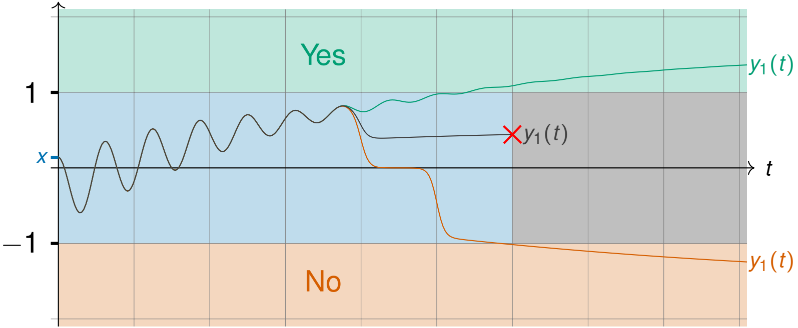


Recap



Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

- analog computability theory
- purely continuous characterization of classical computability

The **complexity** theory of computable functions

Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x

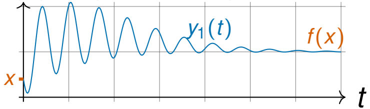
Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x
- GPAC :

Tentative definition

$T(x) = ??$

$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$



Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x
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$T(x, \mu) =$

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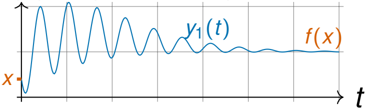

Complexity of analog systems

- Turing machines : $T(x) =$ number of steps to compute on x
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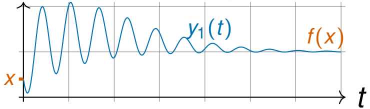
Tentative definition

$T(x,\mu) =$ first time t so that $|y_1(t) - f(x)| \leqslant e^{-\mu}$

$y(0) = (x, 0, \ldots, 0) \qquad y' = p(y)$



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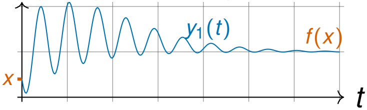
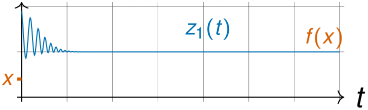


$\rightsquigarrow$

$z(t) = y(e^t)$



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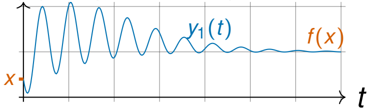

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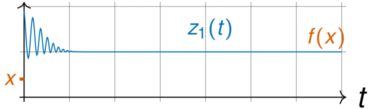
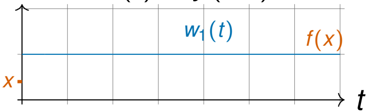


$w(t) = y(e^{e^t})$



Complexity of analog systems

- Turing machines : $T(x)$ = number of steps to compute on x
- GPAC : time contraction problem $\rightarrow$ **open problem**

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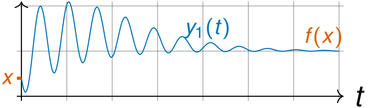

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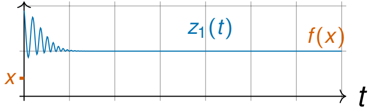

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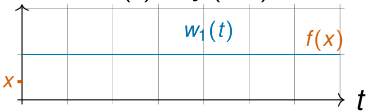
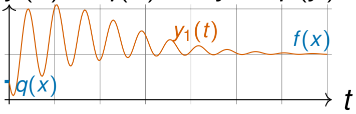


Something is wrong...

All functions have constant time complexity.

Time-space correlation of the GPAC

$y(0) = q(x) \qquad y' = p(y)$



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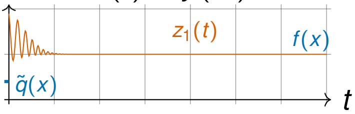
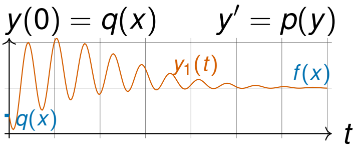


Time-space correlation of the GPAC



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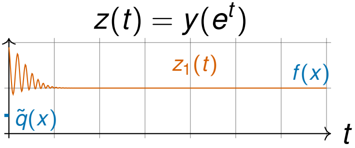
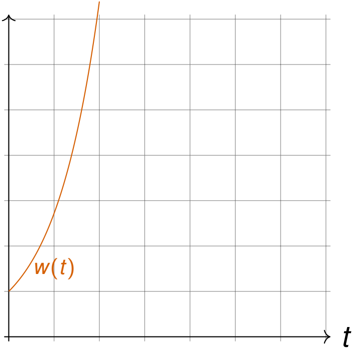


extra component : $w(t) = e^t$



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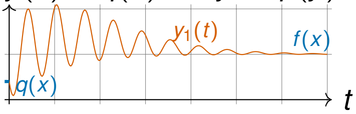

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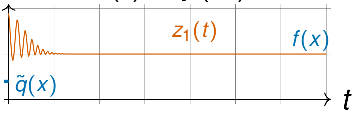

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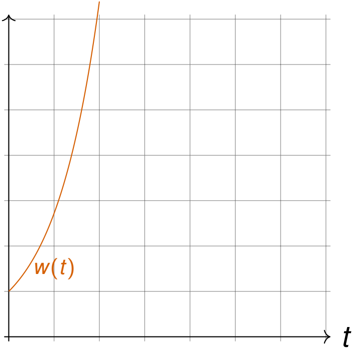


Observation

Time scaling costs "space".

$\rightsquigarrow$

Time complexity for the GPAC must involve time and space !

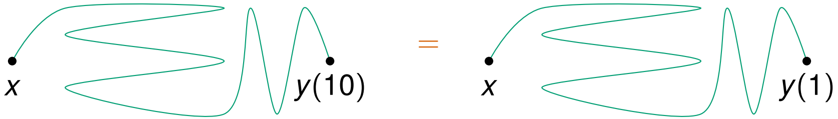
Complexity in the analog world

Complexity measure : length of the curve

Time acceleration : same curve = same complexity !

Complexity in the analog world

Complexity measure : length of the curve



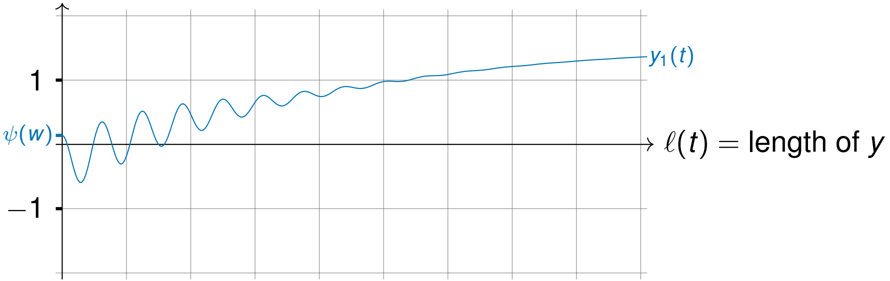
Time acceleration : same curve = same complexity !

Same time, different curves : different complexity !

Characterization of polynomial time

Definition : $\mathcal{L} \in$ ANALOG-PTIME $\Leftrightarrow \exists p$ polynomial, $\forall$ word w

$$y(0) = (\psi(w), |w|, 0, \ldots, 0) \qquad y' = p(y) \qquad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}$$



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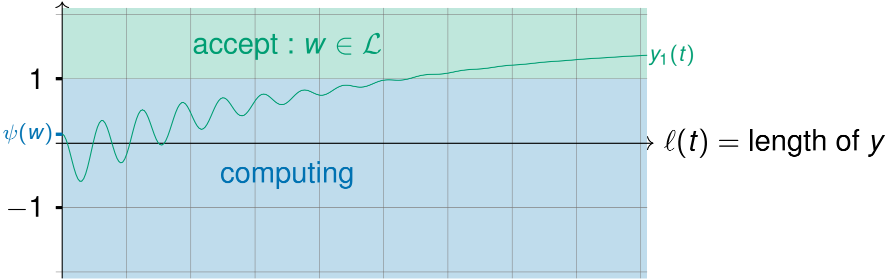

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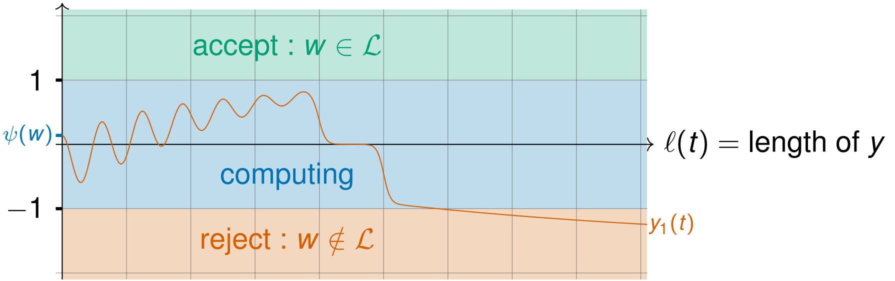

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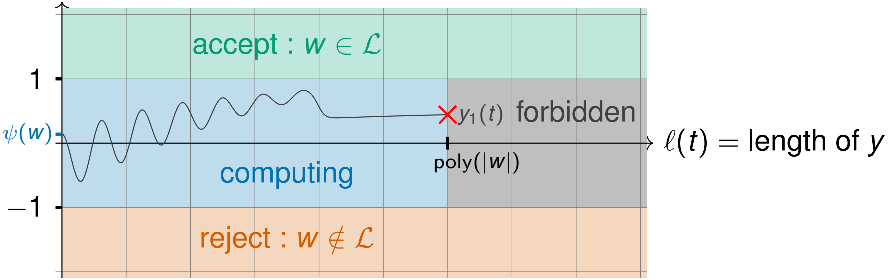

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3. if $\ell(t) \geqslant \text{poly}(|w|)$ then $|y_1(t)| \geqslant 1$

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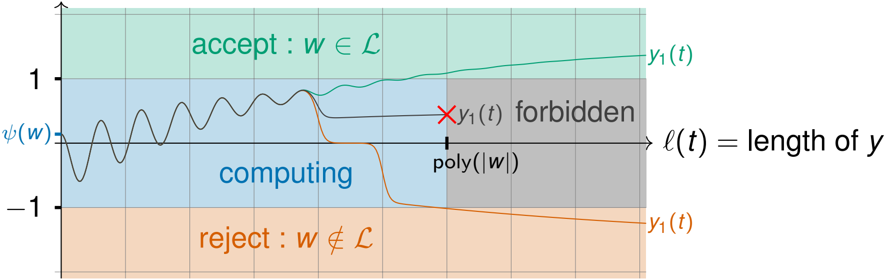
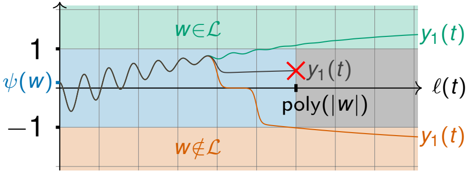


Theorem

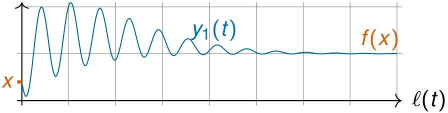
PTIME = ANALOG-PTIME

Summary

ANALOG-PTIME



ANALOG-$P_{\mathbb{R}}$



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- Time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME

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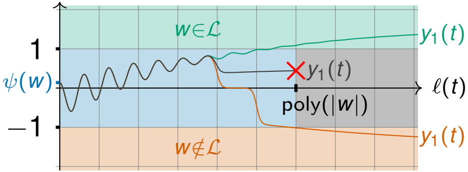

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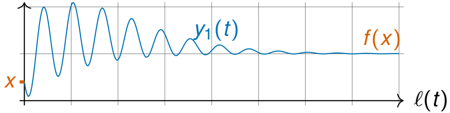

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Chemical Reaction Networks

Chemical Reaction Networks

Definition : a **reaction system** is a finite set of

- molecular species $y_1, \ldots, y_n$
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Example (any resemblance to chemistry is purely coincidental) :

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Theorem (Folklore)

Every polynomial ODE can be rewritten as a quadratic ODE.

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Theorem (Work with François Fages, Guillaume Le Guludec)

Elementary mass-action-law reaction system on finite universes of molecules are Turing-complete under the differential semantics.

Notes :

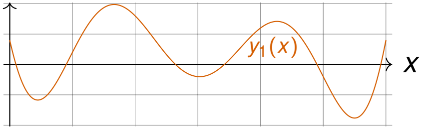
- proof preserves polynomial length
- in fact the following elementary reactions suffice :

$$\varnothing \xrightarrow{k} x \qquad x \xrightarrow{k} x + z \qquad x + y \xrightarrow{k} x + y + z \qquad x + y \xrightarrow{k} \varnothing$$

Universal differential equation

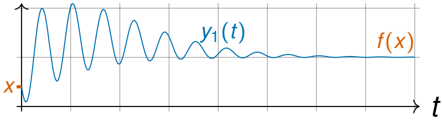
Universal differential equations

Generable functions



subclass of analytic functions

Computable functions



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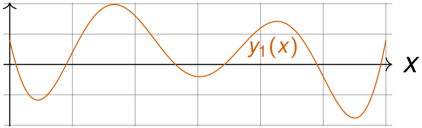

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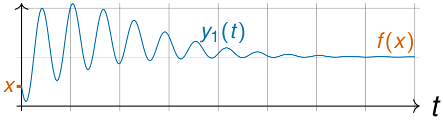

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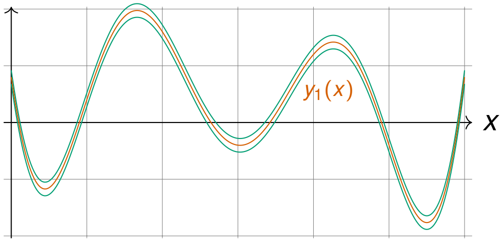
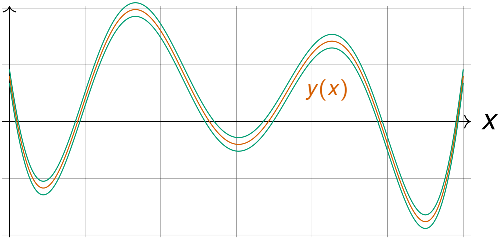


Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

For any continuous functions f and ε, there exists $y:\mathbb{R}\to\mathbb{R}$ solution to

$$\begin{aligned}3y'^4y''y''''^2 &-4y'^4y'''^2y''''+6y'^3y''^2y'''y''''+24y'^2y''^4y'''' \\ &-12y'^3y''y'''^3-29y'^2y''^3y'''^2+12y''^7 \qquad = 0\end{aligned}$$

such that $\forall t\in\mathbb{R}$,

$$|y(t)-f(t)|\leqslant\varepsilon(t).$$

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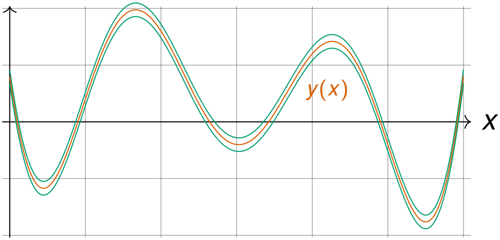

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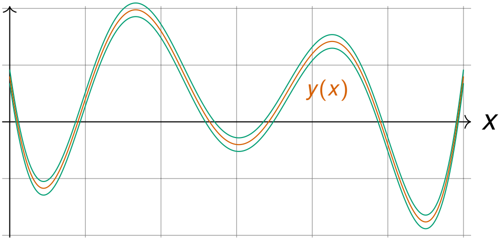

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Problem : this is «weak» result.

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

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- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

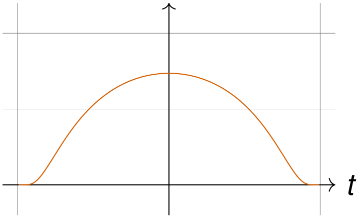
Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

  It satisfies $\quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$



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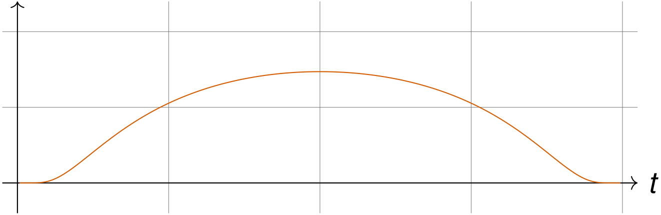
Translation and rescaling :



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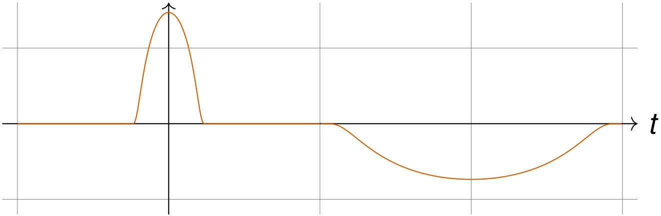

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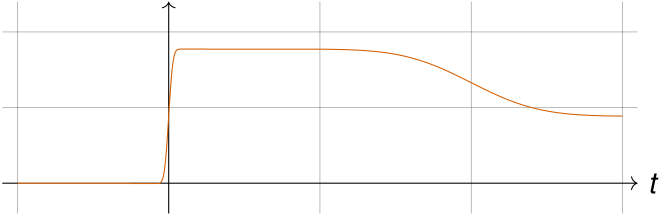

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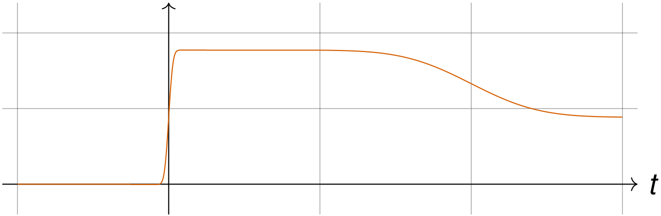
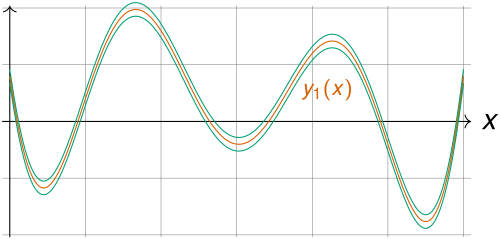


Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

Universal initial value problem (IVP)



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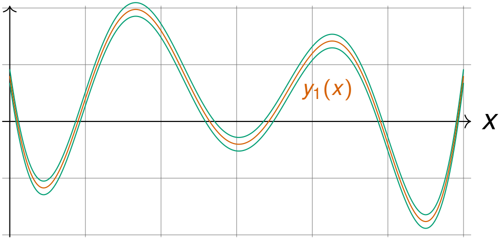

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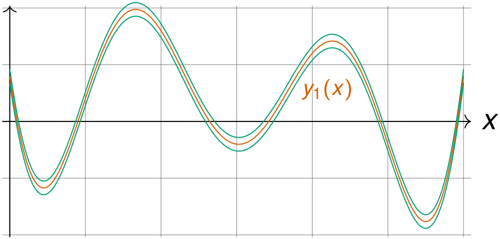

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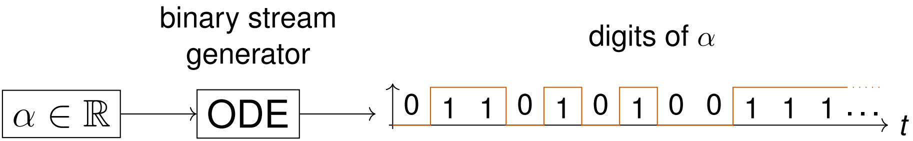
Remark : α is usually transcendental, but computable from f and ε

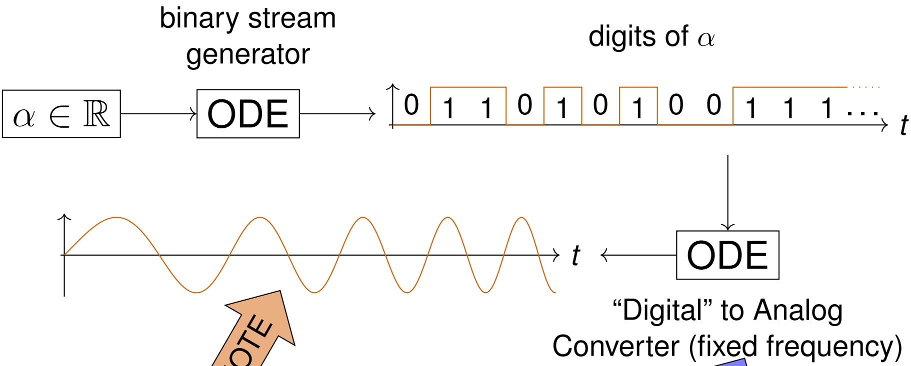
A simplified proof



NOTE

This is the ideal curve, the real one is an approximation of it.

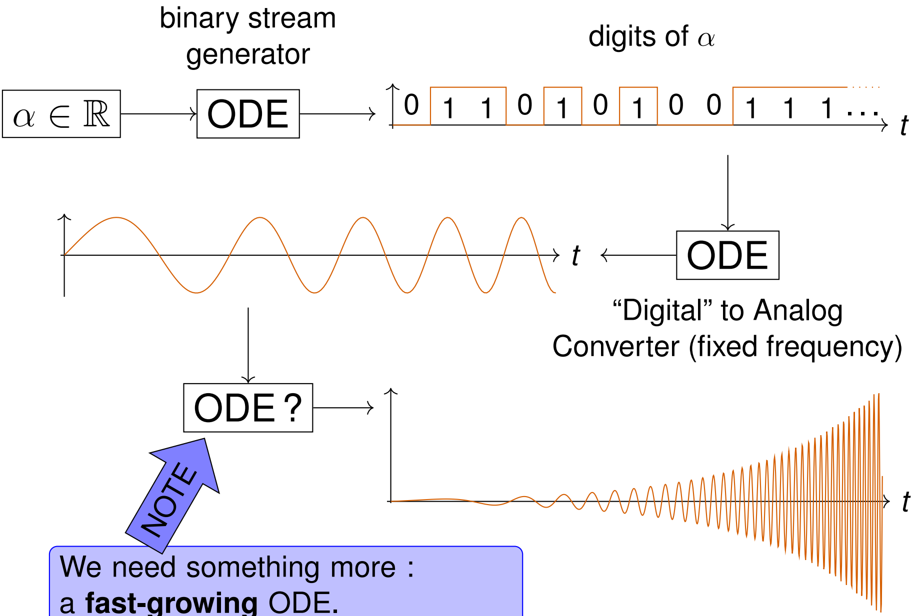
A simplified proof



Approximate Lipschitz and bounded functions with **fixed** precision.

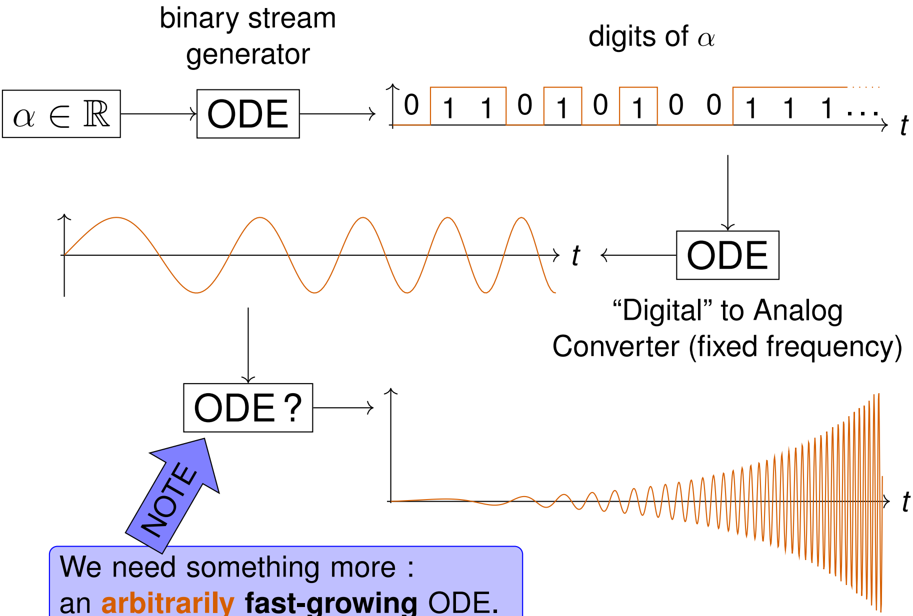
That's the trickiest part.

A simplified proof



We need something more :
a **fast-growing** ODE.

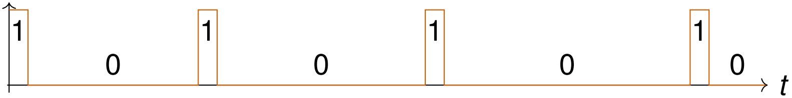
A simplified proof



We need something more :
an **arbitrarily** **fast-growing** ODE.

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



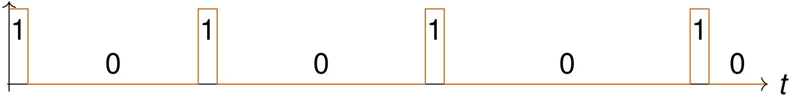
$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2}\tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4,\lambda)} + 4\pi/3))$$

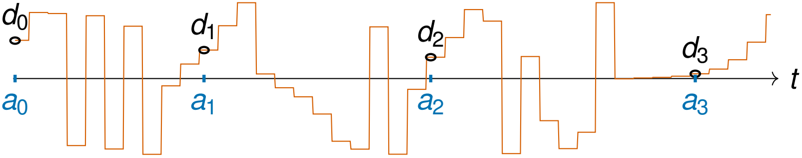
It’s horrible, but generable

round is the mysterious rounding function...

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



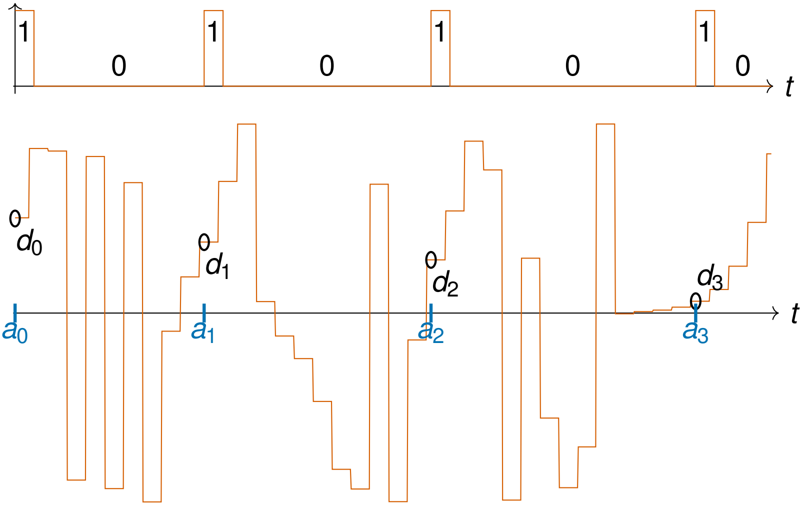


dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j<i} d_j$

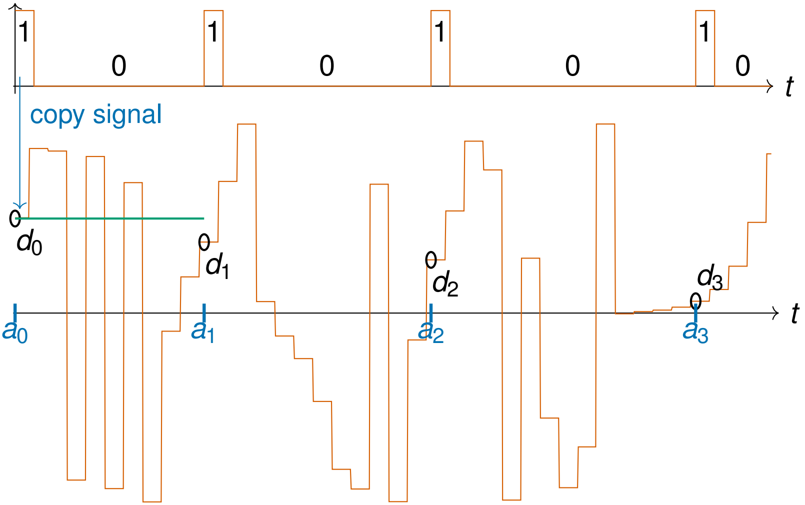
$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4,\gamma)}))$$

round is the mysterious rounding function...

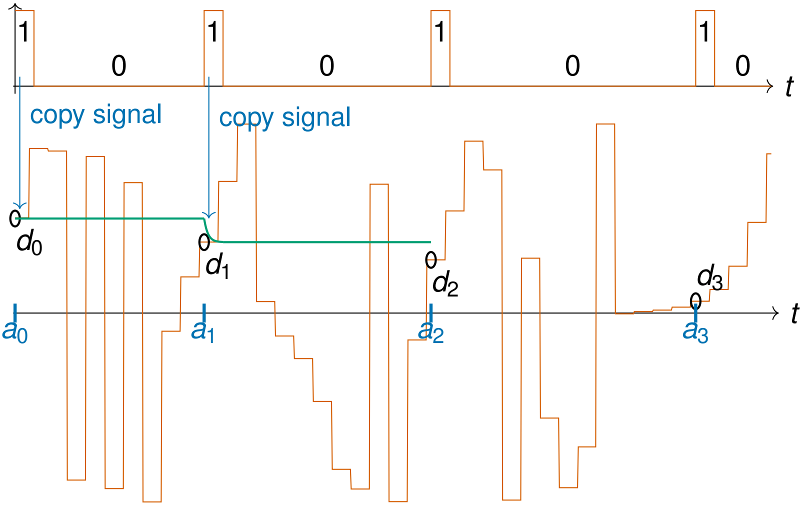
A less simplified proof



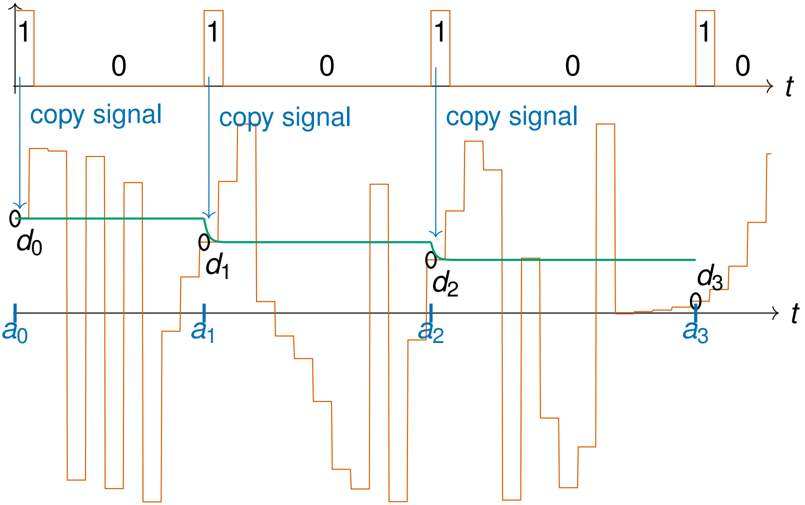
A less simplified proof



A less simplified proof



A less simplified proof



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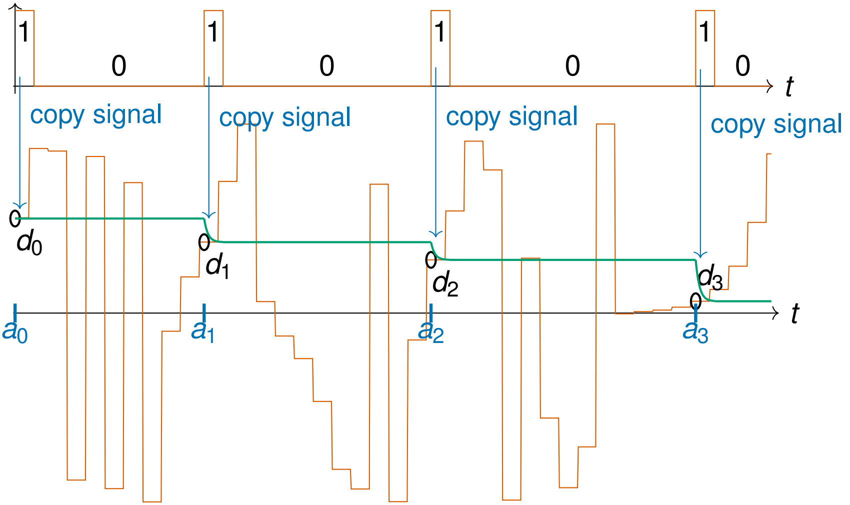

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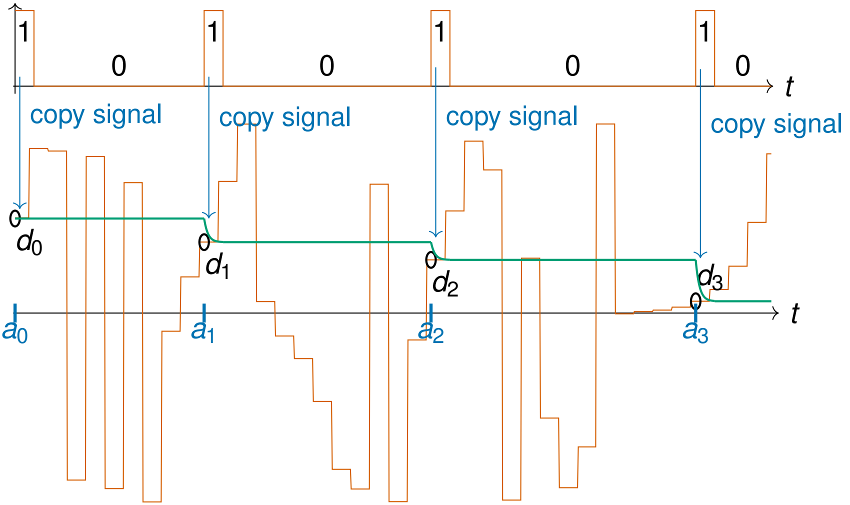
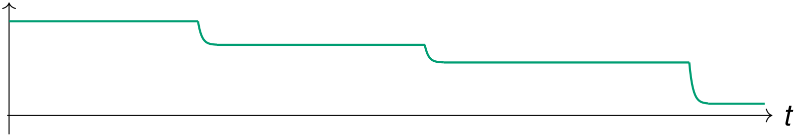


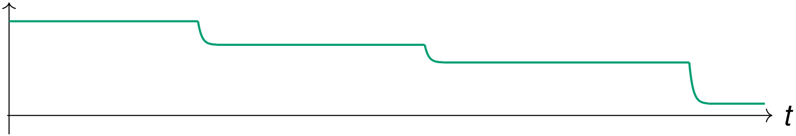
This copy operation is the “non-trivial” part.

A less simplified proof



We can do **almost piecewise constant functions...**

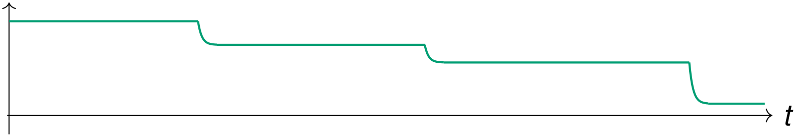
A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
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How do we go to arbitrarily large and growing functions ? Can a polynomial ODE even have arbitrary growth ?

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \qquad \rightsquigarrow \qquad y_1(t) = \exp(t)$$

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$
\begin{array}{lcl}
y_1' = y_1 & \rightsquigarrow & y_1(t) = \exp(t) \\
y_2' = y_1 y_2 & \rightsquigarrow & y_1(t) = \exp(\exp(t))
\end{array}
$$

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

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An old question on growth

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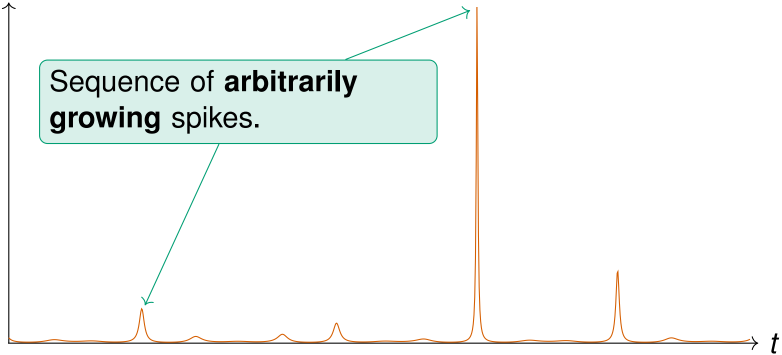
Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

An old question on growth

Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



An old question on growth

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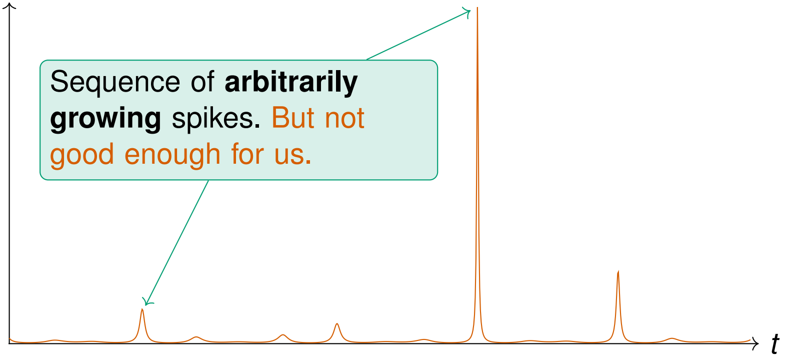

An old question on growth

Theorem

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ *such that for any continuous function* $f : \mathbb{R}_+ \to \mathbb{R}$*, we can find* $\alpha \in \mathbb{R}^d$ *such that*

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

satisfies

$$y_1(t) \geqslant f(t), \qquad \forall t \geqslant 0.$$

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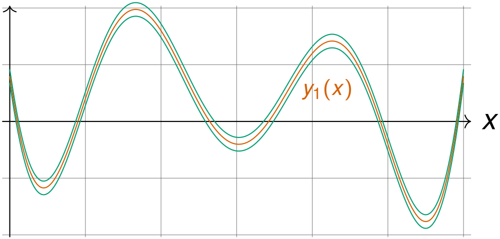
Note : both results require α to be **transcendental.** Conjecture still open for **rational** (or algebraic) coefficients.

Universal initial value problem (IVP)



Theorem

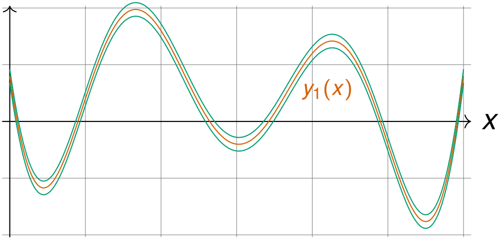
There exists a ***fixed*** *(vector of) polynomial p such that for any continuous functions f and ε, there exists $\alpha \in \mathbb{R}^d$ such that*

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a ***unique solution*** *$y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,*

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Universal initial value problem (IVP)



Notes :

- **system** of ODEs,
- y is analytic,
- we need $d \approx 300$.

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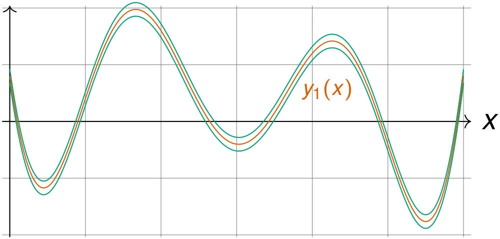

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$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Remark : α is usually transcendental, but computable from f and ε

Future work

Reaction networks :

- chemical
- enzymatic

$$y' = p(y)$$

?

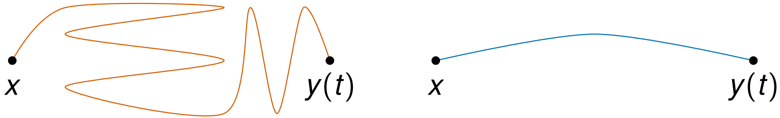
$$y' = p(y) + e(t)$$

- Finer time complexity (linear)
- Nondeterminism
- Robustness
- « Space» complexity
- Other models
- Stochastic

Backup slides

Complexity of solving polynomial ODEs

$$y(0) = x \qquad y'(t) = p(y(t))$$



Complexity of solving polynomial ODEs

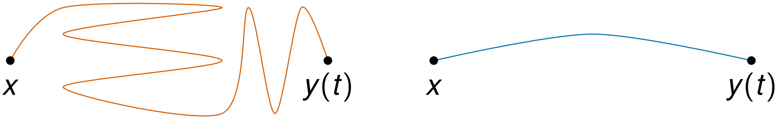
$$y(0) = x \qquad y'(t) = p(y(t))$$

Theorem

If $y(t)$ exists, one can compute p, q such that $\left|\frac{p}{q} - y(t)\right| \leqslant 2^{-n}$ in time

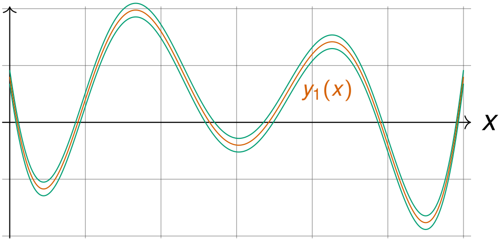
$$\operatorname{poly}(\textit{size of } x \textit{ and } p, n, \ell(t))$$

where $\ell(t) \approx$ length of the curve (between x and $y(t)$)



length of the curve = complexity = ressource

Universal DAE revisited



Theorem

*There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that*

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

*has a **unique analytic solution** and this solution satisfies such that*

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$