Polynomial Invariants for Affine Programs

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Université de Paris, IRIF, CNRS

Does this program halt?

Affine program

$x := 2^{-10}$
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while $y \geqslant x$ do

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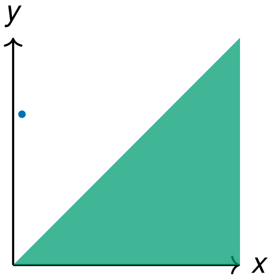

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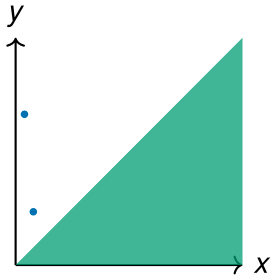

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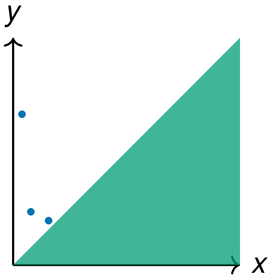

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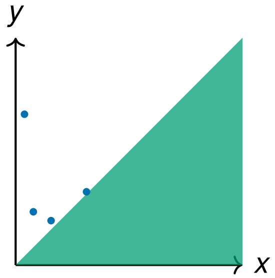

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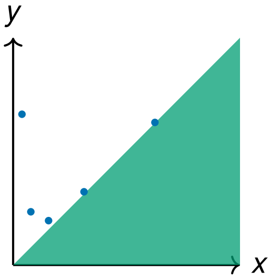

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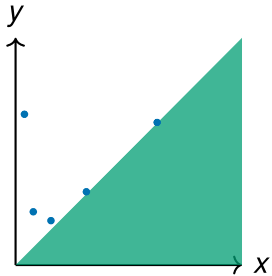
Certificate of non-termination:

$$x^2 y - x^3 = \frac{1023}{1073741824} \qquad (1)$$



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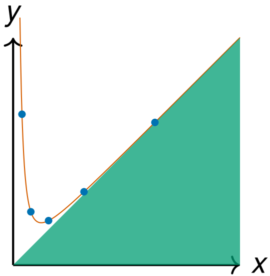

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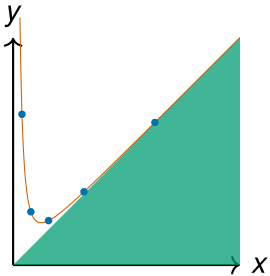

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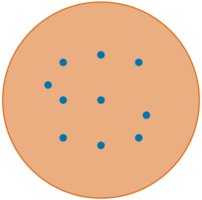
- (1) is an invariant: it holds at every step
- (1) implies the guard is true

Invariants

invariant = overapproximation of the reachable states



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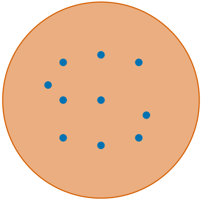
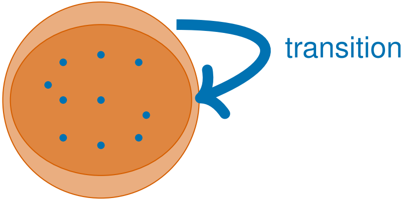
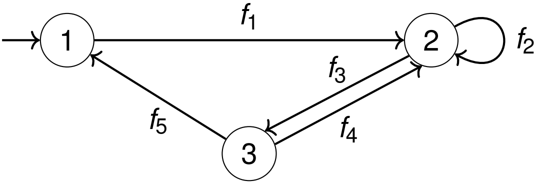


inductive invariant = invariant preserved by the transition relation

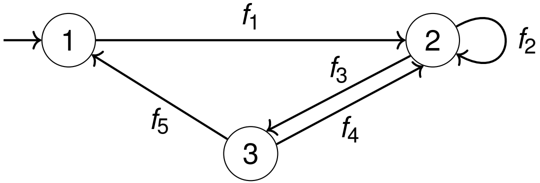


Inductive invariants: example



Inductive invariants: example

x, y, z range over $\mathbb{Q}$ $\qquad$ $f_i : \mathbb{R}^3 \to \mathbb{R}^3$



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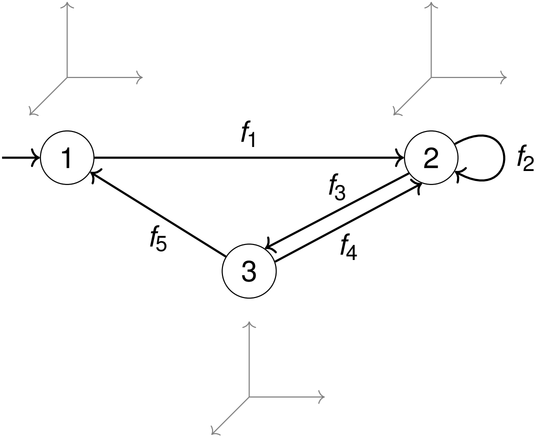

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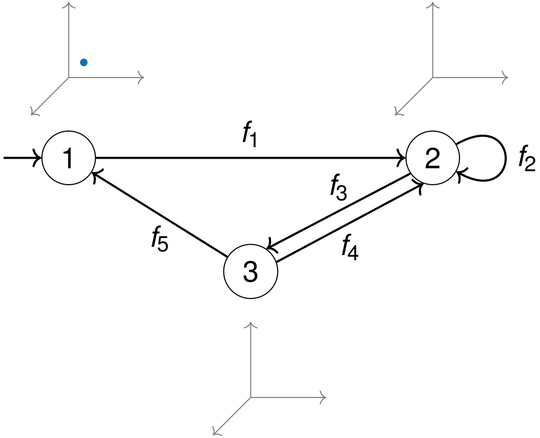

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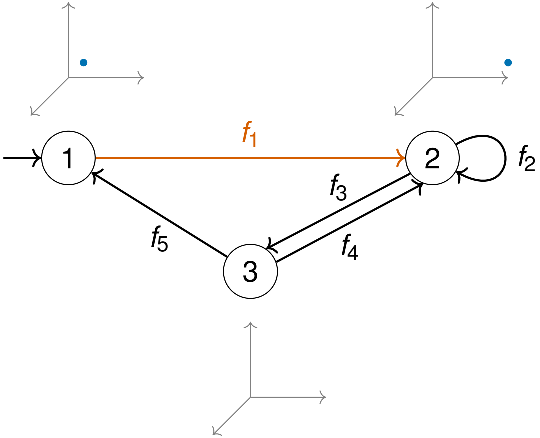

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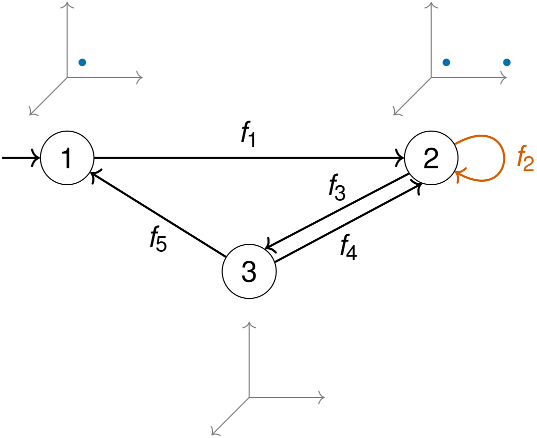

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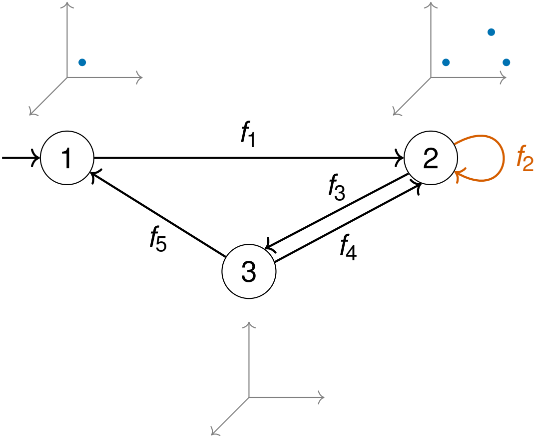

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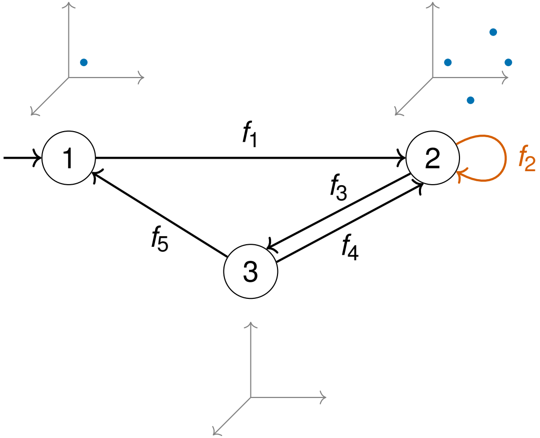

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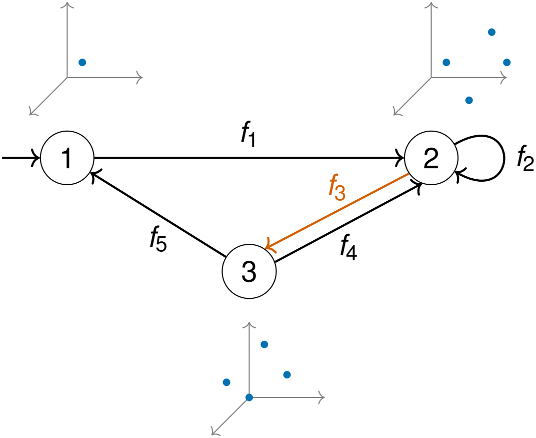

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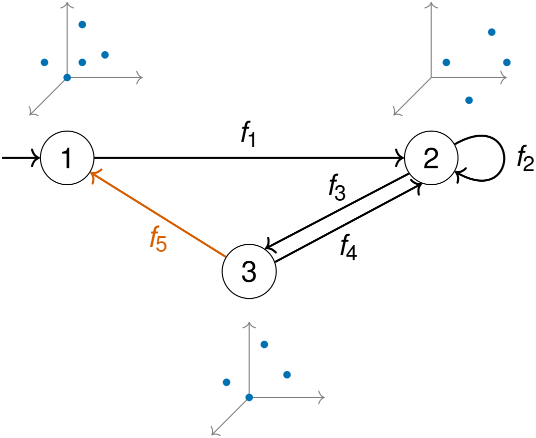

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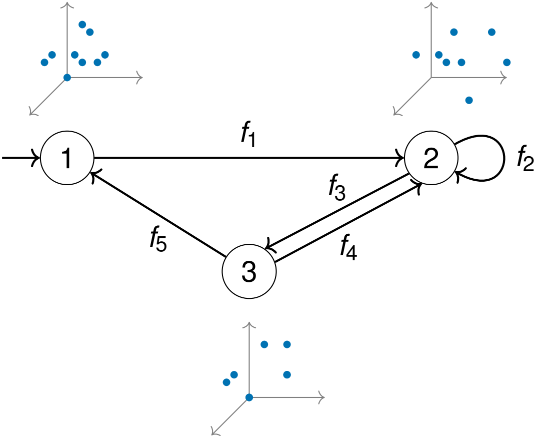

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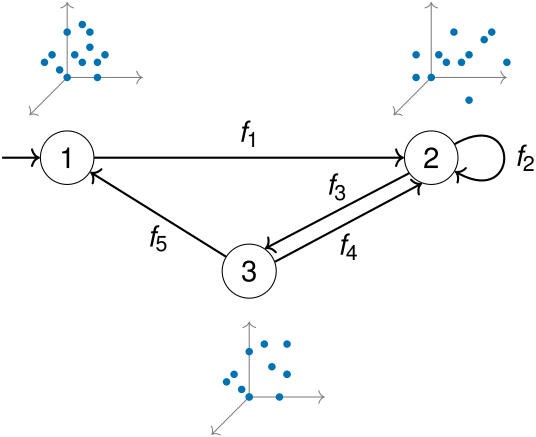

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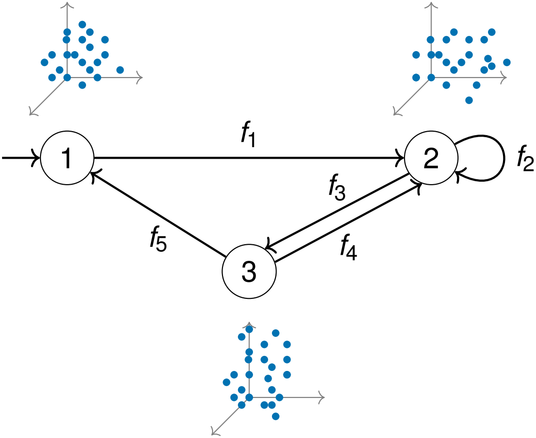

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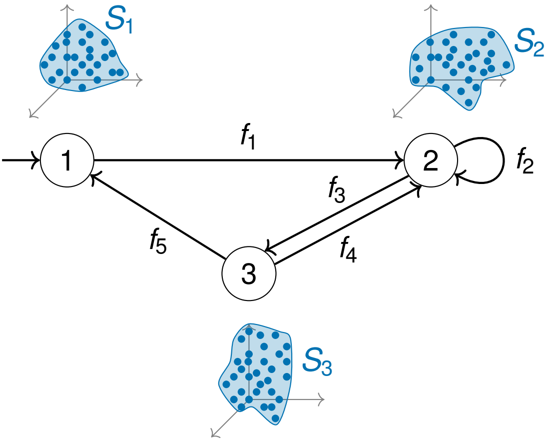

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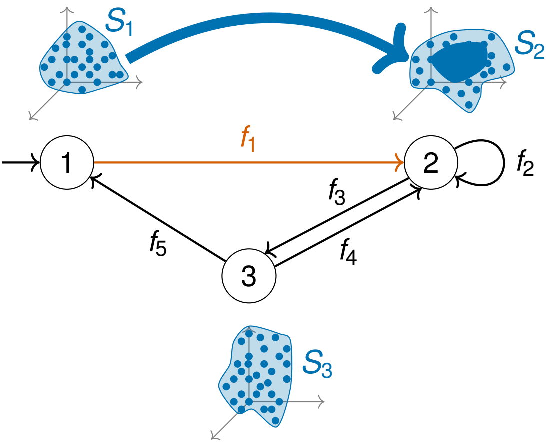

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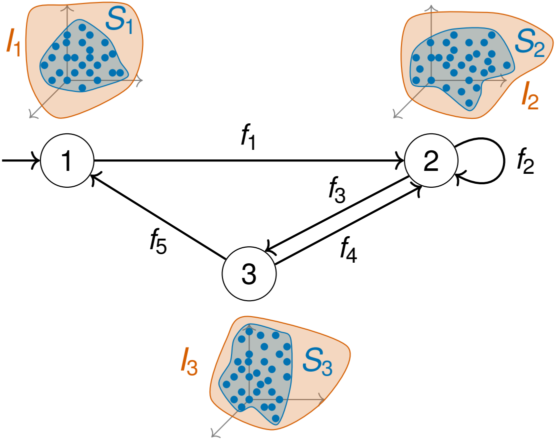

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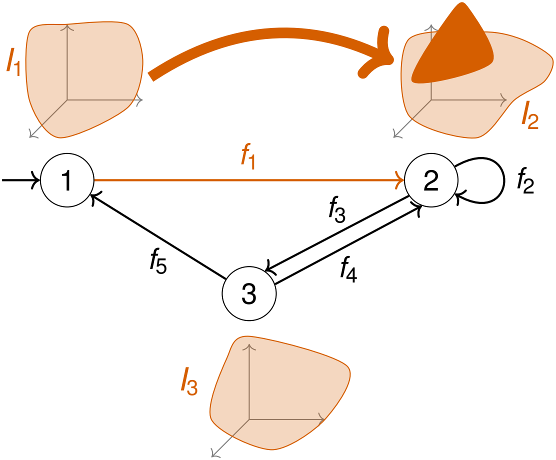

I_1, I_2, I_3 is **NOT** an inductive invariant

Inductive invariants: example

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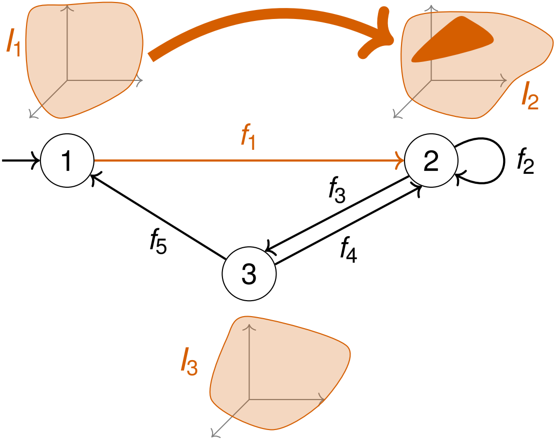
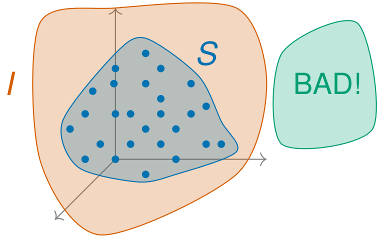


I_1, I_2, I_3 is an inductive invariant

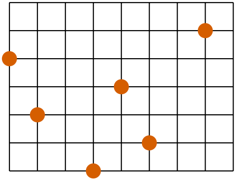
Why Invariants?



The classical approach to the verification of temporal safety properties of programs requires the construction of ***inductive invariants*** *[...].* ***Automation of this construction is the main challenge in program verification.***

D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
Invariant Synthesis for Combined Theories, 2007

Which invariants?



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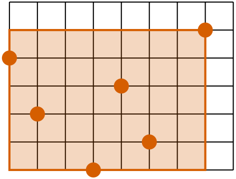

Intervals

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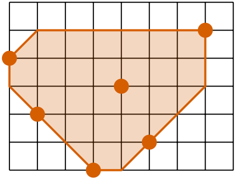
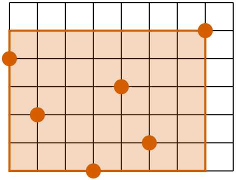


Octagons

$\sqsubseteq$



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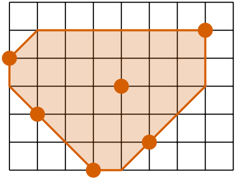

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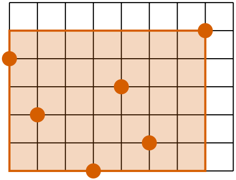
$\rotatebox[origin=c]{90}{$\leq$}$



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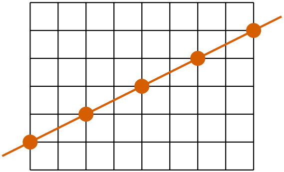
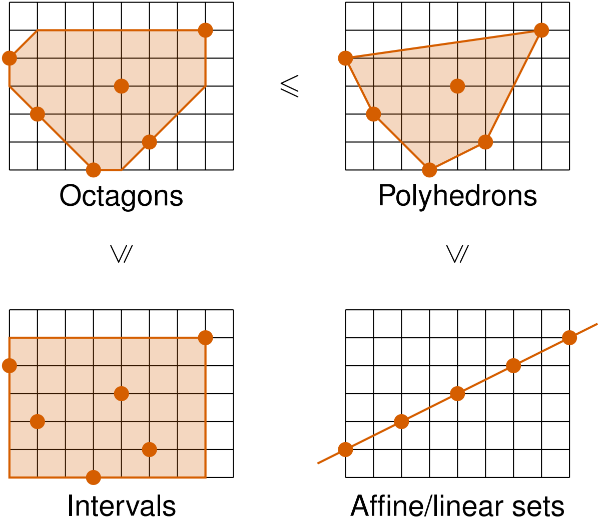


Affine/linear sets

Which invariants?



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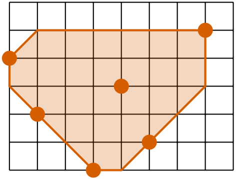

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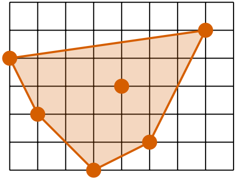

Polyhedrons

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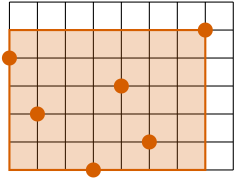

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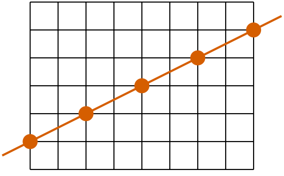

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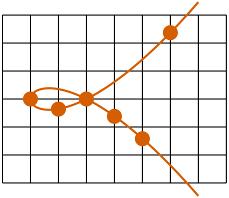
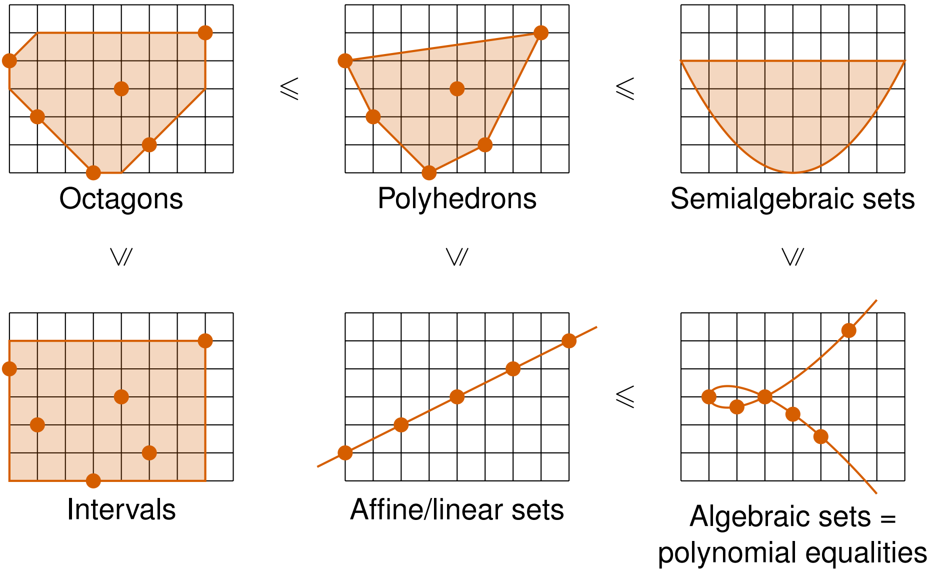
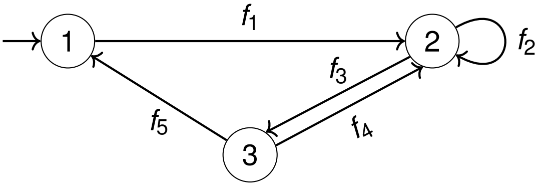


Algebraic sets = polynomial equalities

Which invariants?

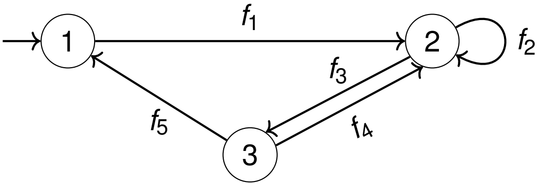


Affine programs



Affine programs

- Nondeterministic branching (no guards)



Affine programs

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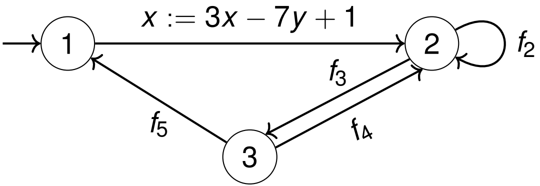

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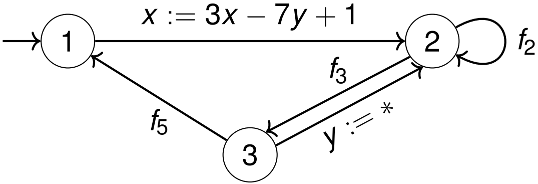

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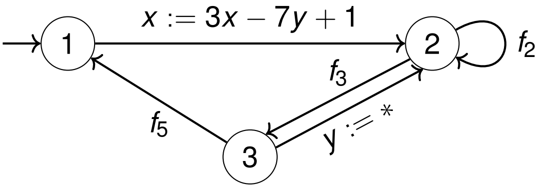

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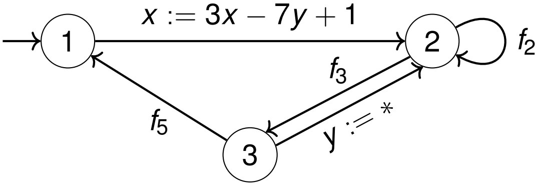

- Can overapproximate complex programs
- Covers existing formalisms:
  probabilistic, quantum, quantitative automata

Affine Relationships Among Variables of a Program*

Michael Karr

Received May 8, 1974

Summary. Several optimizations of programs can be performed when in certain regions of a program equality relationships hold between a linear combination of the variables of the program and a constant. This paper presents a practical approach to detecting these relationships by considering the problem from the viewpoint of linear algebra. Key to the practicality of this approach is an algorithm for the calculation of the "sum" of linear subspaces.

Theorem (Karr 76)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest affine inductive invariant.

Randomized Karr's Algorithm @ POPL 2003

Discovering Affine Equalities Using Random Interpretation

Sumit Gulwani George C. Necula
University of California, Berkeley
{gulwani,necula}@cs.berkeley.edu

ABSTRACT

We present a new polynomial-time randomized algorithm for discovering affine equalities involving variables in a program.

Keywords

Affine Relationships, Linear Equalities, Random Interpretation, Randomized Algorithm

Some polynomial invariants

A Note on Karr's Algorithm

Markus Müller-Olm[1*] and Helmut Seidl[2]

Abstract. We give a simple formulation of Karr's algorithm for computing all affine relationships in affine programs. This simplified algorithm runs in time $\mathcal{O}(nk^3)$ where n is the program size and k is the number of program variables assuming unit cost for arithmetic operations. This improves upon the original formulation by a factor of k. Moreover, our re-formulation avoids exponential growth of the lengths of intermediately occurring numbers (in binary representation) and uses less complicated elementary operations. We also describe a generalization that determines all polynomial relations up to degree d in time $\mathcal{O}(nk^{3d})$.

Theorem (ICALP 2004)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, all its polynomial inductive invariants up to any ***fixed degree*** *d.*

A challenge: finding all polynomial invariants

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Computing polynomial program invariants

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A challenge: finding all polynomial invariants

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Computing polynomial program invariants

Markus Müller-Olm [a,*,1], Helmut Seidl [b]

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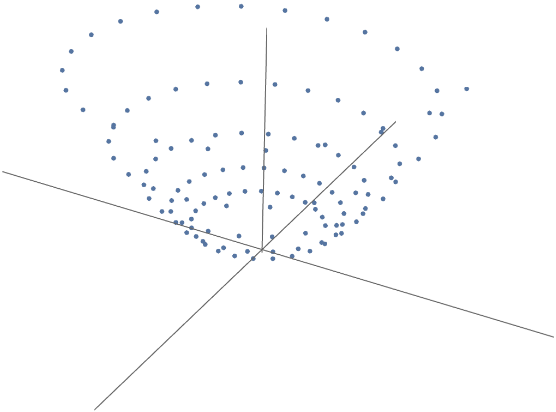
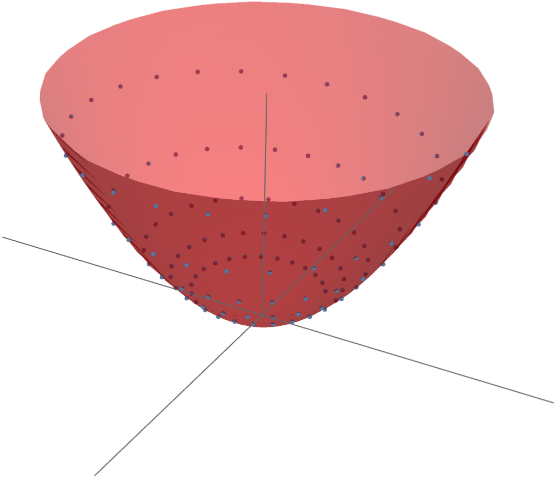
It is a challenging open problem whether or not the set of *all* valid polynomial relations can be computed not just the ones of some given form. It is not

Why fixed degree is not enough



Why fixed degree is not enough

- Paraboloid $z = x^2 + y^2$



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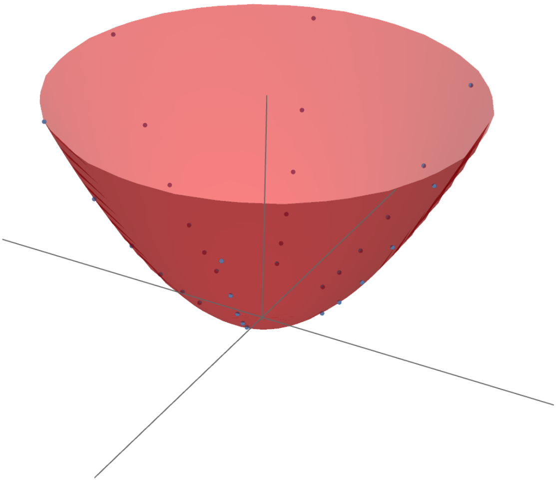

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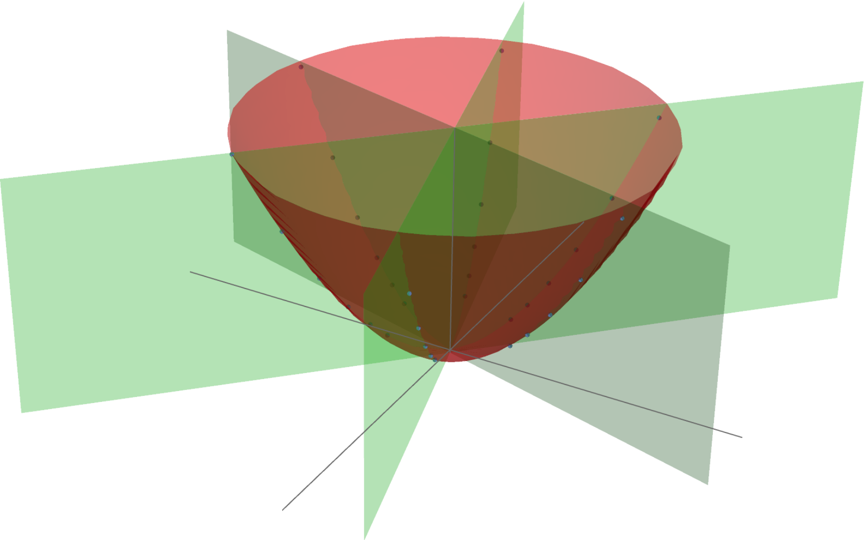
- Paraboloid $z = x^2 + y^2$
- Union of 3 hyperplanes $(x - y)(10y + x)(y + 10x) = 0$



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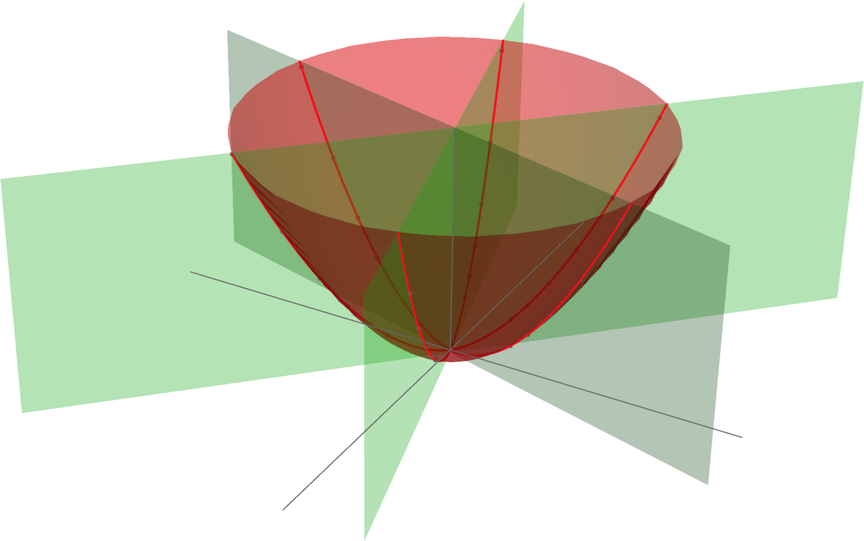

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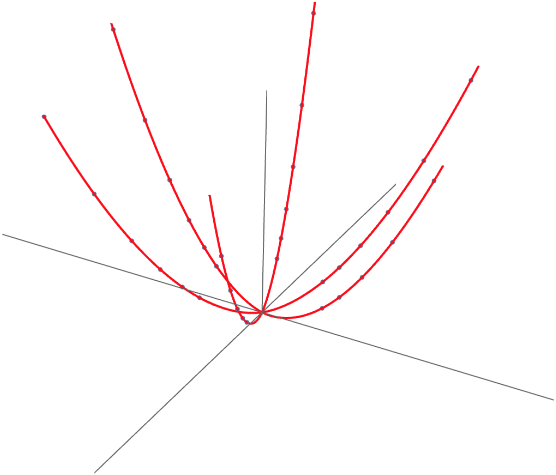

Main result

Theorem

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest polynomial inductive invariant.

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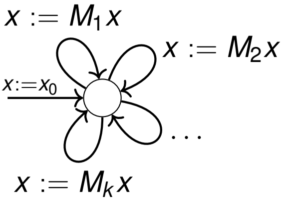
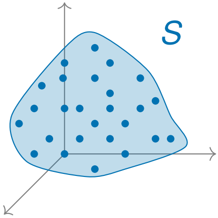
Theorem

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest polynomial inductive invariant.

- strongest polynomial invariant $\iff$ smallest algebraic set
  - algebraic sets = finite $\bigcup$ and $\bigcap$ of polynomial equalities
- Thus our algorithm computes all polynomial relations that always hold among program variables at each program location, in all possible executions of the program
- We can represent this (usually infinite) set of relations using a finite basis of polynomial equalities

At the edge of decidability





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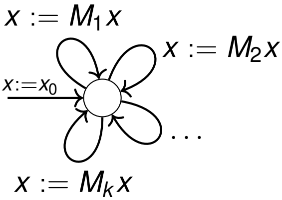
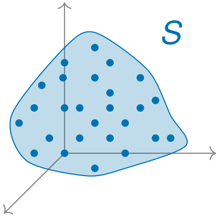




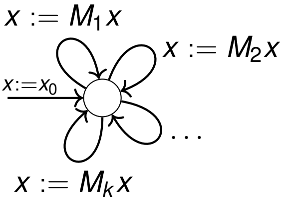
Theorem (Markov 1947*)

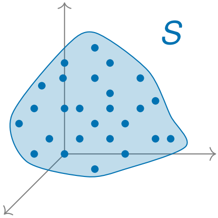
There is a fixed set of 6×6 *integer matrices* $M_1, \dots, M_k$ *such that the reachability problem "*y *is reachable from* x_0*?" is undecidable.*

---

*Original theorems about semigroups, reformulated with affine programs.

At the edge of decidability





Theorem (Markov 1947*)

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Theorem (Paterson 1970*)

The mortality problem "0 is reachable from x_0 with $M_1, \dots, M_k$?" is undecidable for 3×3 *matrices.*

*Original theorems about semigroups, reformulated with affine programs.

Tools

- Algebraic geometry
- Number theory
- Group theory

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- Number theory
- Group theory

Quantum automata and algebraic groups

Harm Derksen[a], Emmanuel Jeandel[b], Pascal Koiran[b,*]

[a]*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, United States*
[b]*Laboratoire de l'Informatique du Parallélisme, Ecole Normale Supérieure de Lyon, 69364, France*

Received 15 September 2003; accepted 1 November 2004

Theorem (Derksen, Jeandel and Koiran, 2004)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$ using only invertible transformations, its strongest polynomial inductive invariant.

Main contribution

Theorem

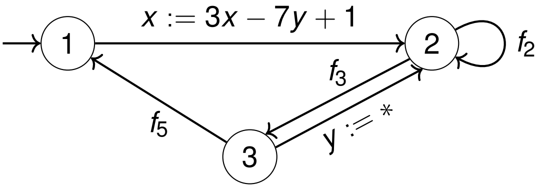
Given a finite set of rational square matrices of the same dimension, we can compute the Zariski closure of the semigroup that they generate.

Corollary

Given an affine program, we can compute for each location the ideal of all polynomial relations that hold at that location.

Summary

- invariant = overapproximation of reachable states
- invariants allow verification of safety properties
- affine program:
  - nondeterministic branching, no guards, affine assignments



Theorem

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest polynomial inductive invariant.

Introduction to Algebraic Geometry (for computer scientists)

Amaury Pouly

Université de Paris, IRIF, CNRS

Overview of this tutorial

A very incomplete introduction to

- Polynomial ideals
- Affine varieties
- Zariski topology
- Constructible sets
- Regular maps

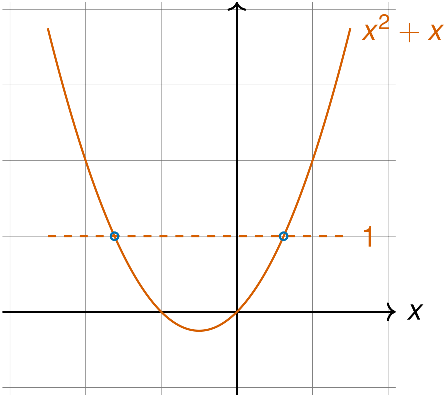
And algorithmic aspects of the above topics.

Everywhere $\mathbb{K}$ is a field, most of the time $\mathbb{K} = \mathbb{C}$.

Motivating examples

Solutions to $x^2 + x = 1$?

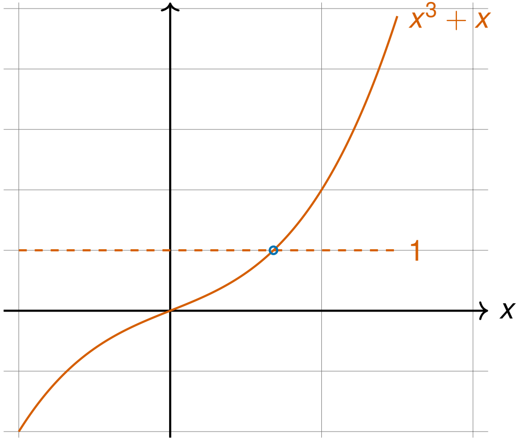
- $S = \left\{-\frac{1}{2} + \frac{1}{2}\sqrt{5}, -\frac{1}{2} - \frac{1}{2}\sqrt{5}\right\}$



Motivating examples

Solutions to $x^3 + x = 1$?

- $S = \left\{ \frac{1}{6}\sqrt[3]{108 + 12\sqrt{93}} - \frac{2}{\sqrt[3]{108+12\sqrt{93}}} \right\}$

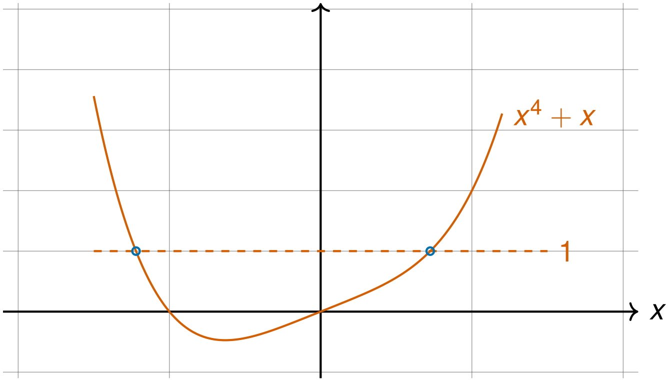


Motivating examples

Solutions to $x^4 + x = 1$?

▶ $S = \{\text{no formula}\}$

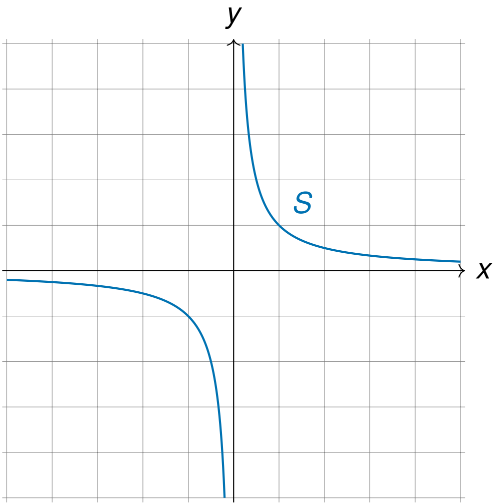
- 2 isolated real roots
- we can approximate them
- algebraic numbers: arithmetic and comparisons are decidable



Solutions to $xy = 1$?

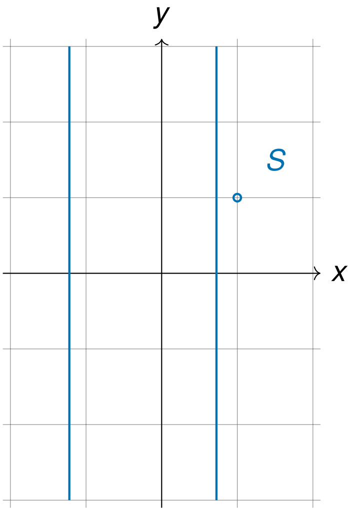
- $S = \left\{ (x, \frac{1}{x}) : x \neq 0 \right\}$

Although we have a formula, the geometry is more interesting.



Motivating examples

Solutions to $((x-1)^2+(y-1)^2)(x^4+x-1)=0$? ► $S=\{$no formula$\}$



No formula in general, but geometry:

- one isolated point
- two infinite curves

Algebraic Geometry is about manipulating those objects, **without** having explicit solutions.

What is Algebraic Geometry?

Study systems of multivariate polynomial equations using abstract algebraic techniques, with applications to geometry.

Examples

$$x^2 + y^2 + z^2 - 1 = 0 \qquad \rightsquigarrow \qquad \text{sphere in } \mathbb{R}^3$$

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The field $\mathbb{K}$ is very important:

- real algebraic geometry: more “intuitive” but more difficult, really requires the study of *semi-algebraic sets*
- mainstream algebraic geometry: $\mathbb{K}$ is algebraically closed†, e.g. $\mathbb{C}$

†$\mathbb{K}$ is algebraically closed if every non-constant polynomial has a root in $\mathbb{K}$.

Polynomial ideals

A set of polynomials $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is an ideal if

- $\forall f, g \in I.\, f + g \in I$ ► I is stable under addition
- $\forall f \in I. \forall g \in \mathbb{K}[x_1, \ldots, x_n] : fg, gf \in I$ ► I absorbs multiplication

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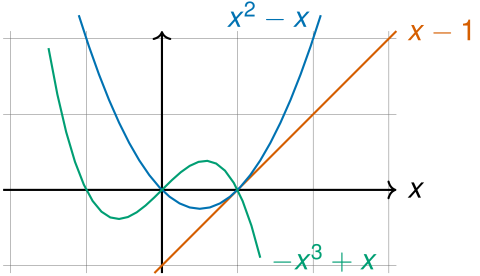
Example: $I = \{p \in \mathbb{K}[x] : p(1) = 0\}$

- if $f(1) = g(1) = 0$ then $(f + g)(1) = f(1) + g(1) = 0$
- if $f(1) = 0$ then for any $g \in \mathbb{K}[x]$, $(fg)(1) = f(1)g(1) = 0$



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Two main ways to create ideals:

- The vanishing polynomials on $S \subseteq \mathbb{K}^n$ is an ideal:

$$I(S) := \{f \in \mathbb{K}[x_1, \ldots, x_n] : \forall x \in S.\, f(x) = 0\}$$

Remark: I is inclusion reversing, $S \subseteq S' \;\Rightarrow\; I(S) \supseteq I(S')$

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- The ideal generated by $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$ is

$$\begin{aligned}\langle f_1, \ldots, f_k \rangle &:= \text{smallest ideal containing } f_1, \ldots, f_k \\ &:= \{p_1 f_1 + \cdots + p_k f_k : p_1, \ldots, p_k \in \mathbb{K}[x_1, \ldots, x_n]\}\end{aligned}$$

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Example: $\{p \in \mathbb{K}[x] : p(1) = 0\} = I(\{1\}) = \langle x - 1 \rangle$.

Polynomial ideals: important facts

A set of polynomials $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is an ideal if

- $\forall f, g \in I : f + g \in I$ ► I is stable under addition
- $\forall f \in I, \forall g \in \mathbb{K}[x_1, \ldots, x_n] : fg, gf \in I$ ► I absorbs multiplication

Theorem (Hilbert's basis theorem)

For any field $\mathbb{K}$, $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian: any chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes: $\exists k \in \mathbb{N}$ such that $I_k = I_{k+1} = I_{k+2} = \cdots$.

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Corollary

Every polynomial ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is finitely generated:

$$\exists f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n] \text{ such that } I = \langle f_1, \ldots, f_k \rangle.$$

We can represent ideals by a finite set of generators.

Polynomial ideals: important operations

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- quotient: $(I : J) := \{r : rJ \subseteq I\}$

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Remark: $I \cup J$ is not an ideal but $I + J = \langle I \cup J \rangle$

All these operations are effective.

Algebraic sets

Algebraic set: set of the common zeroes of polynomials

$$V(S) = \{x \in \mathbb{K}^n : \forall p \in S.\, p(x) = 0\} \text{ where } S \subseteq \mathbb{K}[x_1, \dots, x_n]$$

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Examples

- $\{(x, y) \in \mathbb{K}^2 : y = x^2\}$
- $\{(x, y, z) \in \mathbb{K}^3 : x = y^2 \wedge y = z\}$
- $\mathbb{K}^n = \{x \in \mathbb{K}^n : 0 = 0\}$
- $\varnothing = \{x \in \mathbb{K}^n : 1 = 0\}$
- $\{a\} = \{x : x_1 - a_1 = \ldots = x_n - a_n = 0\}$

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For arbitrary S, $V(S) = V(I)$ where $I = \langle S \rangle$ is the ideal generated by S.

$\rightsquigarrow$ Always take S to be an ideal, this gives us a finite representation of algebraic sets.

Algebraic sets / Zariski topology

Algebraic set: set of the common zeroes of an ideal $I \subseteq \mathbb{K}[x_1, \dots, x_n]$

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Basic properties:

- stable under finite unions: $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
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Zariski topology: the closed set are the algebraic sets

Examples

- $\{(x, y) \in \mathbb{K}^2 : y = x^2\} = V(y - x^2)$ is closed
- $\{(x, y) \in \mathbb{K}^2 : y \neq x^2\} = \mathbb{K}^2 \setminus V(y - x^2)$ is open

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$Y \subseteq \mathbb{K}^n$ is irreducible if it is not the union of two proper closed subsets.

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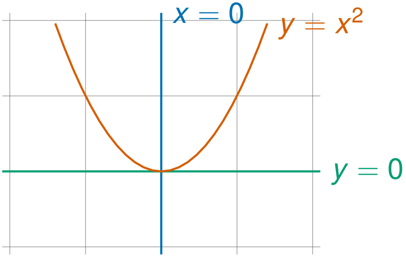
- $\{(x, y) : xy = 0\}$ is reducible: $\{(x, y) : x = 0\} \cup \{(x, y) : y = 0\}$



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Theorem

Any algebraic set can be written as the finite union of irreducible algebraic sets.

Ascending/Descending chains

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Remark: the last fact comes from the notion of dimension of an algebraic set. It is geometrically “what one would expect”: a curve has dimension 1, a hypersurface $n - 1$, the whole space n.

Affine varieties

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⚠ The term affine variety is ambiguous, it can mean

- algebraic set
- irreducible algebraic set

In this lecture

affine variety = algebraic set

Zariski topology / Zariski closure

Let $X \subseteq \mathbb{K}^n$ be a variety. The Zariski topology on X has as closed sets the subvarieties of X: the sets $A \subseteq X$ that are varieties in $\mathbb{K}^n$.

Examples

- $X = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$ is closed in $\mathbb{R}^3$
- $S = X \cap \left\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\right\}$ is closed in X

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Examples

- $\overline{[-1, 1]}^{\mathbb{R}} = \mathbb{R}$
- $\overline{\mathbb{N}}^{\mathbb{R}} = \mathbb{R}$
- $\overline{\left\{(x, y) \in \mathbb{R}^2 : x \geqslant 0 \wedge x = y^2\right\}}^{\mathbb{R}} = \left\{(x, y) \in \mathbb{R}^2 : x = y^2\right\}$

A quick summary of what we have seen so far

- ideal: set of polynomials, stable under $+$, absorbing $\times$
- algebraic set: common zeroes of a set of polynomials/ideal
- irreducible set: not the union of two proper algebraic subsets
- affine variety: (irreducible) algebraic set (author dependent)
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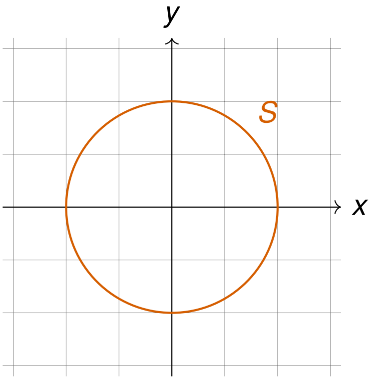
- effective operations: union and intersection of closed sets

Quantifier Elimination (QE): $\mathbb{R}$ vs $\mathbb{C}$

Let $S = \{(x, y) \in \mathbb{K}^2 : x^2 + y^2 = 1\}$.



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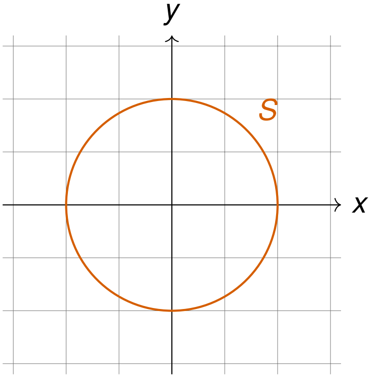

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Two very different behaviors:

- For $\mathbb{K} = \mathbb{R}$:
$$S' = [-1, 1] = \left\{x \in \mathbb{R} : x^2 \leqslant 1\right\}$$
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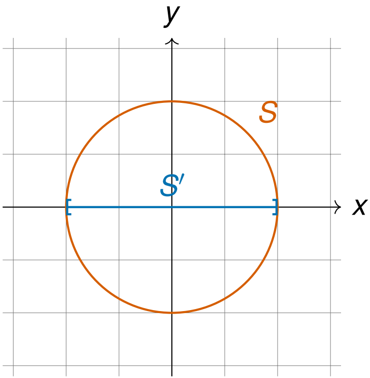

In $\mathbb{R}$ we need to introduce inequalities.

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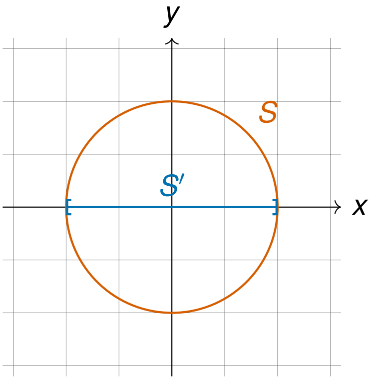


Theorem (QE over $\mathbb{R}$)

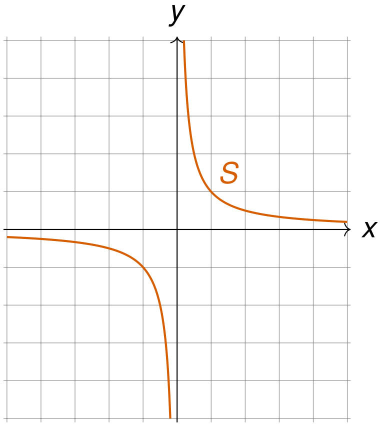
$(\mathbb{R}, +, \times, 0, 1, \leqslant)$ *admits QE.*

Theorem (QE over $\mathbb{C}$)

$(\mathbb{C}, +, \times, 0, 1, =)$ *admits QE.*

Definable/Constructible sets: motivation

$S = \left\{(x, y) \in \mathbb{R}^2 : xy = 1\right\}$ ►
variety/closed set



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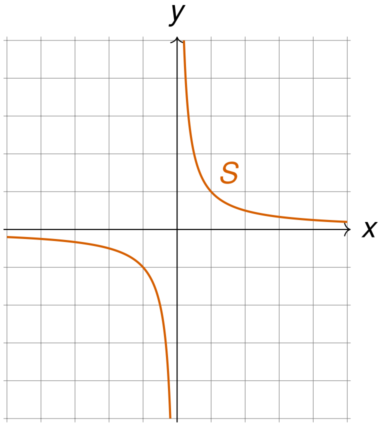
$p(x, y) = x$
▶ "nice" function (polynomial)



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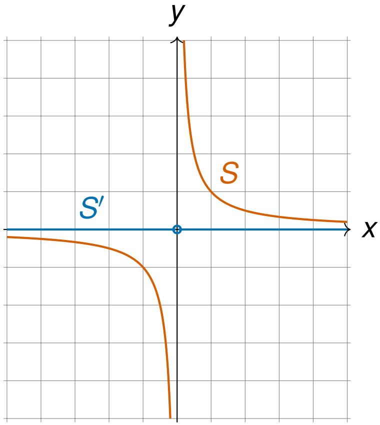

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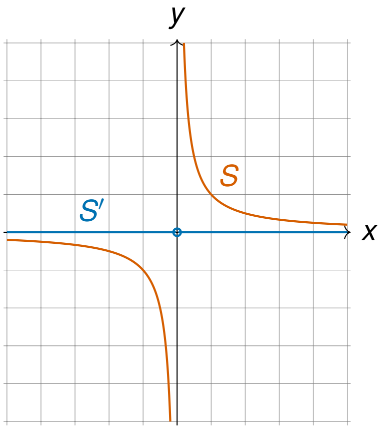

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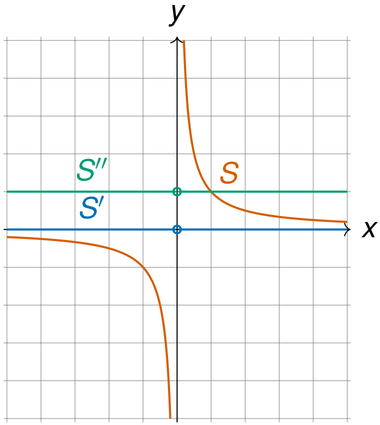

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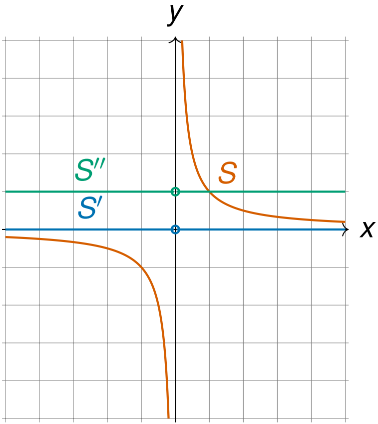

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We need something more general than varieties: the above sets are

- definable: $\{x \in \mathbb{K}^n : \phi(x)\}$
- constructible: intersections/unions of open/closed sets

Constructible/Definable sets

A set S is definable if $S = \{x \in \mathbb{K}^n : \phi(x)\}$ for ϕ first-order formula[‡].

Examples

- any variety: $\phi(x) \equiv \bigwedge_i p_i(x) = 0$
- any open set: $\phi(x) \equiv \neg \bigwedge_i p_i(x) = 0$
- $S = \{(x, y) : y = 1 \wedge x \neq 0\}$
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The constructible sets are all Boolean combinations (including complementation) of Zariski closed sets.

Examples

- any closed or open set
- $S = \{(x, 1) : x \neq 0\} = \{(x, y) : y = 1\} \cap \{(x, y) : x = 0\}^{\complement}$
- $S = \{(x, y) : x = 0\}^{\complement} \cup \{(0, 0)\}$

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Theorem (Consequence of quantifier elimination)

For $\mathbb{K} = \mathbb{C}$, the constructible sets are exactly the definable sets.

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In this lecture

We use constructible sets over $\mathbb{C}$ everywhere.

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Theorem (Chevalley)

The image of a constructible set under a polynomial map is constructible.

(also follows from quantifier elimination)

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- Zariski closure: $\overline{S}$ where S constructible
  Common use: $\overline{p(S)}$ where S constructible and p polynomial

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Constructible sets: decomposition and application

The constructible sets are all Boolean combinations (including complementation) of Zariski closed sets.

Lemma

If X is constructible then $\exists A_1, \ldots, A_k$ irreducible and $B_1, \ldots, B_k$ closed,

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Exercice: if A irreducible, B closed and $A \setminus B \neq \varnothing$ then $\overline{A \setminus B} = A$

$A = (A \setminus B) \cup (A \cap B) \rightsquigarrow A = \overline{A \setminus B} \cup (A \cap B)$ then use irreducibility

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Application: Zariski closure of a constructible set X

$$\overline{X} = \overline{\bigcup_{i=1}^{k} A_i \setminus B_i} = \bigcup_{i=1}^{k} \overline{A_i \setminus B_i} = \bigcup_{i=1}^{k} \overline{A_i} \qquad \text{assuming } A_i \setminus B_i \neq \varnothing$$

Summary

- ideal: set of polynomials, stable under $+$, absorbing $\times$
- algebraic set: common zeroes of a set of polynomials/ideal
- irreducible set: not the union of two proper algebraic subsets
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- constructible set: Boolean combinations of closed sets
- definable set: first-order definable with equality
- effective operations: union, intersection, complementation, first-order definition, image under polynomial map, Zariski closure