On the Computation of the Zariski Closure of Finitely Generated Groups of Matrices

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Motivation

Affine program

 $x := 2^{-10}$ y := 1while $y \ge x$ do $\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

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Certificate of non-termination:

$$x^2y - x^3 = \frac{1023}{1073741824} \tag{1}$$

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Computing such invariants reduces to computing the **Zariski closure** of a semigroup of matrices.



Quantum automata

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A (measure once) quantum finite automaton (QFA):

- Σ: finite alphabet,
- ▶ $s \in \mathbb{C}^n$: vector of unit norm,
- $X_a \in \mathbb{C}^{n \times n}$: unitary transition matrix for each $a \in \Sigma$,
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Value of a word $w \in \Sigma^*$:

 $\operatorname{Val}_{\mathcal{A}}(w) = \| PX_w s \|_2^2$ where $X_w = X_{w_|w|} \cdots X_{w_1}$

Interpretation: the probability of observing the quantum state in acceptance space after having applied the operator sequence X_{w_1} to $X_{w_{|w|}}$ to the initial quantum states.

Given a QFA \mathcal{A} and a threshold λ :

Emptiness Problem

 $\exists w \in \Sigma^* \text{ such that } \mathsf{Val}_\mathcal{A}(w) \geq \lambda$?

Undecidable*: proof by reduction from PCP.

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Strict Emptiness Problem

 $\exists w \in \Sigma^* \text{ such that } \operatorname{Val}_{\mathcal{A}}(w) > \lambda ?$

Decidable*: reduces to computing the **Zariski closure** of a group of matrices

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The Problem

The Zariski topology

Algebraic set: set of common zeroes of a collection S of polynomials in $\mathbb{A}[x_1, \ldots, x_n]$:

$$V(S) = \{x \in \mathbb{A}^n : \forall p \in S, p(x) = 0\}$$



Hilbert's basis theorem: for any S, there exists S' finite s.t. V(S) = V(S')

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Zariski closure of a set X is the smallest algebraic set \overline{X} that contains X.



Zariski closure of finitely generated matrix semigroups

Given $A_1, \ldots, A_k \in \mathbb{A}^{n \times n}$, consider

 $\langle A_1, \ldots, A_k \rangle$ = semigroup generated by the A_i .

Problem: compute $\overline{\langle A_1, \ldots, A_k \rangle}$.

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- $\overline{\langle A_1, \ldots, A_k \rangle}$ is an algebraic set, the output of the algorithm is a finite set of polynomials,
- view $\mathbb{A}^{n \times n}$ as \mathbb{A}^{n^2} to make sense of the closure.

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- view $\mathbb{A}^{n \times n}$ as \mathbb{A}^{n^2} to make sense of the closure.

Example:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \rightsquigarrow \qquad \langle S, T \rangle = \mathsf{SL}_2(\mathbb{Z})$$

then

$$\overline{\langle S,T\rangle} = SL_2(\overline{\mathbb{Z}}) = SL_2(\mathbb{A}) = \left\{ M \in \mathbb{A}^{n \times n} : \det(M) = 1 \right\}.$$

Given a finite set $S \subseteq \mathbb{A}^{n \times n}$ and $d \in \mathbb{N}$, define the "degree-*d* closure" as the smallest algebraic set that contains $\langle S \rangle$ and is defined by polynomials of total degree at most *d*.

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Theorem (Karr, 1974; Müller-Olm and Seidl, 2004)

There is an algorithm that computes, given S and d, the degree-d closure of $\langle S \rangle$, in time $O(|S| \cdot (n^2 + 1)^{3d})$.

There is even a randomized algorithm by Gulwani and Necula (2003).

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Remarks:

- most applications do not need the closure: a sufficiently good approximation is sufficient
- surely one can obtain an upper bound on d?



Paraboloid

$$z = x^2 + y^2$$



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- Paraboloid
- Union of 3 hyperplanes

 $z = x^{2} + y^{2}$ (x - y)(10y + x)(y + 10x) = 0



History of the problem (cont)

Quantum automata and algebraic groups

Harm Derksen^a, Emmanuel Jeandel^b, Pascal Koiran^{b,*}

^aDepartment of Mathematics, University of Michigan, Ann Arbor, MI 48109, United States ^bLaboratoire de l'Informatique du Parallélisme, Ecole Normale Supérieure de Lyon, 69364, France

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There is an algorithm that computes $\overline{\langle S \rangle}$ given a finite set S of matrices.

None of these algorithms puts a bound on the degree of the closure!

We obtain a degree bound for invertible matrices:

Theorem

Given a finite set S of invertible matrices of dimension n, the algebraic group $G := \overline{\langle S \rangle}$ can be defined with equations of degree at most septuply exponential in n.

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Let $n \in \mathbb{N}$ and let $S \subseteq GL_n(\mathbb{Q})$ be a finite set of matrices whose entries have height at most h. Then the Zariski closure of the group generated by S can be represented by finitely many polynomials of degree at most $(\log h)^{2^{|S|^{exp^4}(\operatorname{poly}(n))}}$ with coefficients in \mathbb{Q} , forming a basis of the vanishing ideal of the group generated by S. Furthermore, if G contains only semisimple elements then the degree can be bounded by $(\log h)^{2^{|S|^{2^{poly}(n)}}}$.
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Corollary

The algebraic closure of a finitely genrated matrix group is computable in elementary (octuply exponential) time.

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$$\mathsf{GL}_n(\mathbb{A}) = \{ (M, y) \in \mathbb{A}^{n^2+1} : \mathsf{det}(M) \cdot y = 1 \}$$

$$\mathsf{SL}_n(\mathbb{A}) = \{(M, y) \in \mathbb{A}^{n^2+1} : \mathsf{det}(M) \cdot y = 1, \mathsf{det}(M) = 1\}$$

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Key fact: if $S \subseteq GL_n(\mathbb{A})$ then $\overline{\langle S \rangle}$ is an algebraic group

We analyse the structure of algebraic groups that come from finitely generated groups.

- Ehud Hrushovski (2002): computable (no degree bound)
- Ruyong Feng (2015): sextuply exponential
- Mengxiao Sun (2018): triple exponential
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We use many ideas from the above papers to prove our result. Future work: use the techniques of Amzallag, Minchenko and Pogudin to reduce our bound

Remarks on lower bounds

A difficulty in the proof is that the degree bound must depend on the height of the entries:

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$$(2^p)^1 \cdot \left(\frac{1}{2}\right)^p = 1.$$

Therefore any polynomial that vanishes on $\langle A \rangle$ must also vanish on

$$\{\mathsf{diag}(x,y):xy^p=1\}$$

and thus be of degree at least $1 + p \ge \log(h)$.

Conclusion:

- even in dimension 2, the degree can be arbitrarily large and depends on the height.
- the exponential "lower bound" of Amzallag, Minchenko, Pogudin probably also works in our case

The proof yields a potentially useful result on chains of algebraic groups:

Theorem

Let $n \in \mathbb{N}$, k be a number field, and $G_i = \overline{\langle S_i \rangle}$ for $S_i \subseteq GL_n(k)$, $1 \le i \le \ell$, be such that $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_\ell$. Then $\ell \le \exp\left(\operatorname{poly}([k : \mathbb{Q}]) \exp^3(\operatorname{poly}(n))\right)$,

and $\ell \leq 2^{\text{poly}(n[k:\mathbb{Q}])}$ if each G_i consists only of semisimple elements.

This may be useful to analyse the running time of algorithms.

Motivation:

- certifying non-termination of linear loops
- analysing quantum automata

Problem: compute the Zariski closure of a finitely generated group of matrices

- computable
- ▶ we obtained a septuly exponential bound on the degree of the closure

Future work:

- ▶ improve bound using ideas from differential Galois group algorithms
- study special classes of groups
- extend to semigroups

G has a normal subgroup of finite index H:

Good properties

- we know |G/H|,
- ▶ we have degree bounds on *H*



G has a normal subgroup of finite index *H*: *G* is the union of |G/H| copies of *H*

 \sim we can write equations for *G* from that of *H* and |G/H|.

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The quotient G/U is an algebraic group consisting only of semisimple elements.

We can use this to reduce to the case of semisimple matrices!



- $\rightarrow U \lhd G$
- $\rightarrow \phi_U(G) \cong G/U$ semisimple
- \rightarrow Bound on U
- $\rightarrow~$ Bound on the degree of equations defining ϕ_U [Feng'15]





- \rightarrow Pistil *P* of $\phi_U(G)$
- $\rightarrow \phi_U(G) \lhd (\phi_U(G) \cap P) \lhd \phi_U(G)$
- $\rightarrow \phi_U(G) \cap P$ finite index in $\phi_U(G)$
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- \rightarrow *H* finite index in *G*
- $\rightarrow H/U$ commutative

Strict Emptiness

Given a QFA \mathcal{A} and a threshold λ :

$$\exists w \in \Sigma^* \text{ s.t. } Val_{\mathcal{A}}(w) > \lambda ?$$

Note: if such a word exists, it is a finite certificate. Enumerate!

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Observation:

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Finite certificate: $\overline{\mathcal{X}}$ can be finitely represented and is computable. [Derksen et al.'05]

At the edge of decidability





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There is a fixed finite set of 6×6 integer matrices S such that the problem of deciding whether $A \in \langle S \rangle$ for a given A is undecidable.

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Theorem (Paterson 1970)

The problem of deciding, given M_1, \ldots, M_k , whether $0 \in \langle M_1, \ldots, M_k \rangle$ is undecidable for 3×3 matrices.



Nondeterministic branching (no guards)



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- All assignments are affine


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Can overapproximate complex programs

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- Can overapproximate complex programs
- Covers existing formalisms: probabilistic, quantum, quantitative automata

invariant = overapproximation of the reachable states



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inductive invariant = invariant preserved by the transition relation





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x, y, z range over \mathbb{Q}

$$f_i: \mathbb{R}^3 \to \mathbb{R}^3$$



 S_1, S_2, S_3 are the reachable states

x, y, z range over \mathbb{Q}

 $f_i: \mathbb{R}^3 \to \mathbb{R}^3$



 S_1, S_2, S_3 is also an inductive invariant

x, y, z range over \mathbb{Q}

$$f_i: \mathbb{R}^3 \to \mathbb{R}^3$$



 I_1, I_2, I_3 is an invariant

x, y, z range over \mathbb{Q}

$$f_i: \mathbb{R}^3 \to \mathbb{R}^3$$



 I_1, I_2, I_3 is **NOT** an inductive invariant

x, y, z range over \mathbb{Q}

 $f_i: \mathbb{R}^3 \to \mathbb{R}^3$



 I_1, I_2, I_3 is an inductive invariant



The classical approach to the verification of temporal safety properties of programs requires the construction of inductive invariants [...]. Automation of this construction is the main challenge in program verification.

> D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko Invariant Synthesis for Combined Theories, 2007

Which invariants?

