On the complexity of some reachability problems

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Joint work with H. Bazille, O. Bournez, W. Gomaa, D. Graça, J. Ouakine, J. Worrel, J. Sousa-Pinto

10 february 2016

Outline

Piecewise Affine Systems

Polynomial Initial Value Problem

Linear hybrid automata

Piecewise Affine System (1)

General Model

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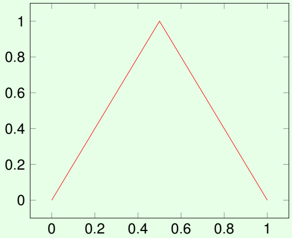
- $\mathbb{K} = [0, 1]$: continuous bounded case $\to$ Our case
- $\mathbb{K} = \mathbb{R}$: continuous unbounded case $\to$ Similar*ish* to $[0, 1]$?

Piecewise Affine System (2)

f continuous

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$



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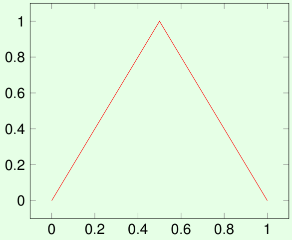

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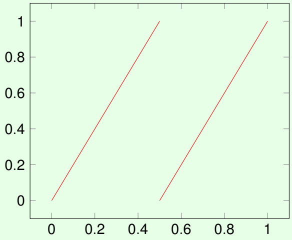

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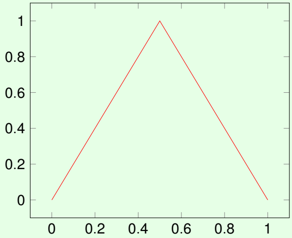

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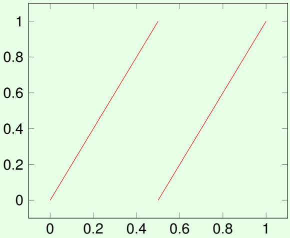

$\rightarrow$ Our case

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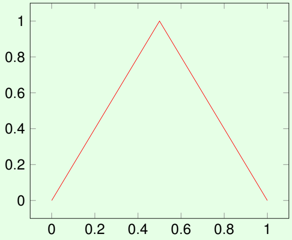

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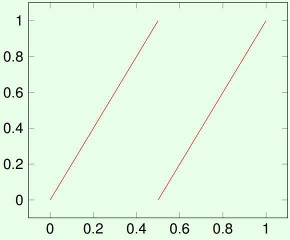

$\rightarrow$ Quite different

Example

Function

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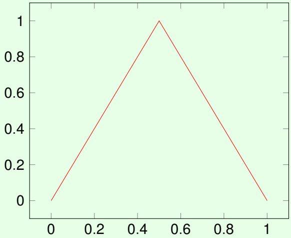

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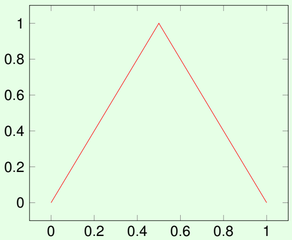
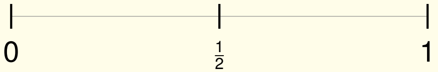


Trajectory



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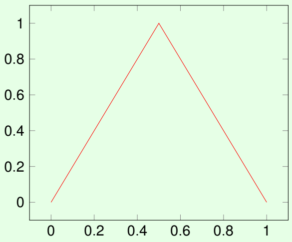

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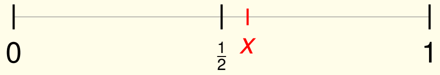

$$x = 0.5625$$

Example

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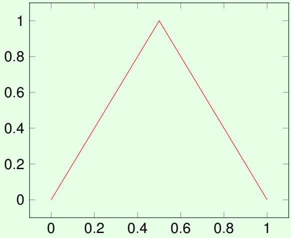

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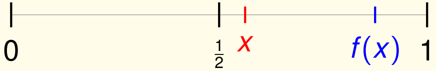

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$$f(x) = 0.875$$

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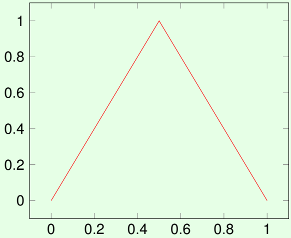

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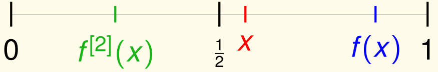

$$x = 0.5625$$
$$f(x) = 0.875$$
$$f^{[2]}(x) = 0.25$$

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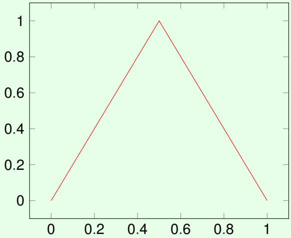

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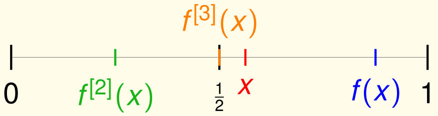

$$x = 0.5625$$
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$$f^{[2]}(x) = 0.25$$
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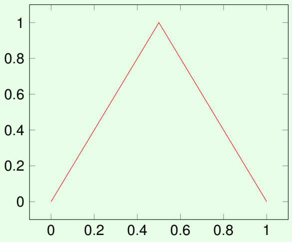

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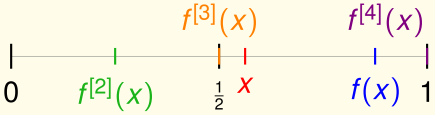

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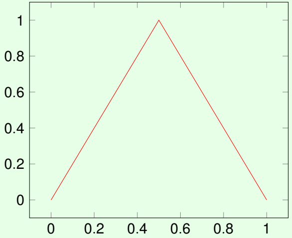

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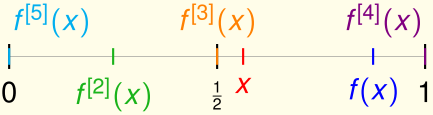

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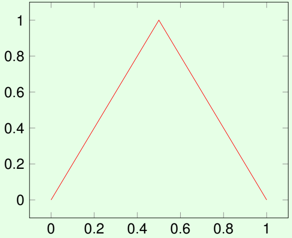

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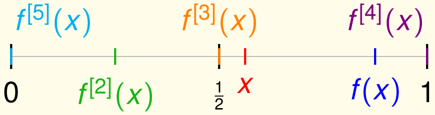

$$\begin{aligned} x &= 0.5625 \\ f(x) &= 0.875 \\ f^{[2]}(x) &= 0.25 \\ f^{[3]}(x) &= 0.5 \\ f^{[4]}(x) &= 1 \\ f^{[n]}(x) &= 0 \qquad n \geqslant 5 \end{aligned}$$

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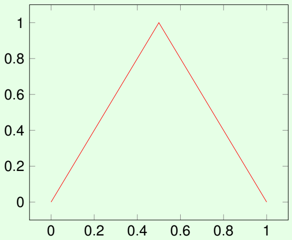
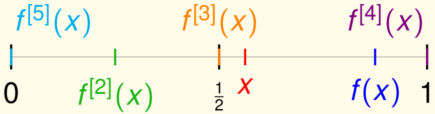


Remark

Trajectory depends on the binary expansion of x

Trajectory



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Existings Results

Problem: `REACH-REGION`

- **Input:** $f : [0,1]^d \to [0,1]^d$ continuous, piecewise affine

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Problem: `REACH-REGION`

- **Input:** $f : [0,1]^d \rightarrow [0,1]^d$ continuous, piecewise affine
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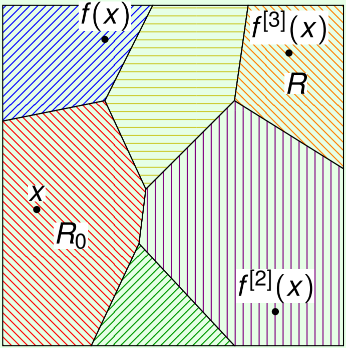

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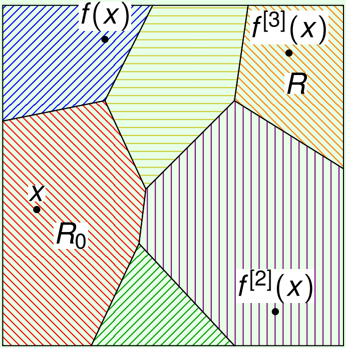

Theorem (Koiran, Cosnard, Garzon)

REACH−REGION is undecidable for $d \geqslant 2$

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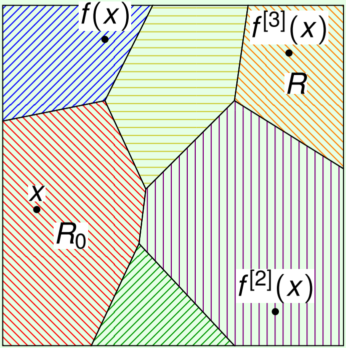

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Proof (Idea)

Simulate a Turing Machine and reduce from halting problem.

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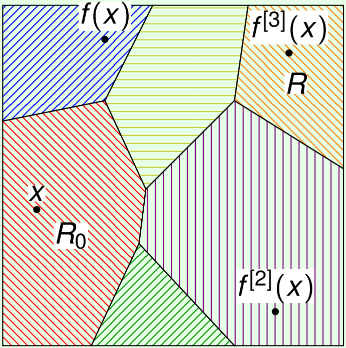

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Open Problem

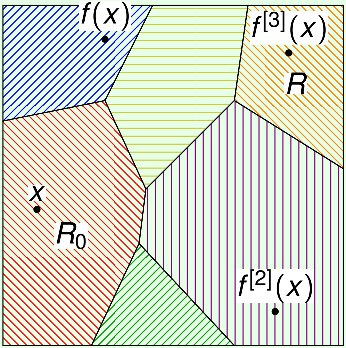
Decidability for $d = 1$, even for two intervals.

Existings Results

Problem: CONTROL−REGION

- **Input:** $f : [0,1]^d \to [0,1]^d$ continuous, piecewise affine
- **Input:** R_0, R: convex regions of $[0,1]^d$
- **Question:** $\forall x \in R_0, \exists t \in \mathbb{N}, f^{[t]}(x) \in R$?

Example



Theorem (Blondel, Bournez, Koiran, Tsitsiklis)

CONTROL−REGION is undecidable for $d \geqslant 2$

Proof (Idea)

Harder simulation of a Turing Machine

Open Problem

Decidability for $d = 1$, even for two intervals.

Our Results

Problem: REACH-REGION-TIME

- **Input:** $f : [0,1]^d \rightarrow [0,1]^d$ continuous, piecewise affine

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Complexity for $d = 1$.

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Theorem

CONTROL-REGION-TIME is coNP-complete for $d \geqslant 2$

Open Problem

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REACH−REGION−TIME is in NP

Proof.

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Works because:

Every satisfiable rational linear system $Ax \leqslant b$ has a rational solution of polynomial size.

REACH-REGION-TIME is NP-hard

Reduce from:

Problem SUBSET-SUM

- **Input:** a goal $B \in \mathbb{N}$ and integers $A_1, \ldots, A_n \in \mathbb{N}$
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- Still reduction is very tricky

REACH-REGION-TIME is NP-hard (proof idea)

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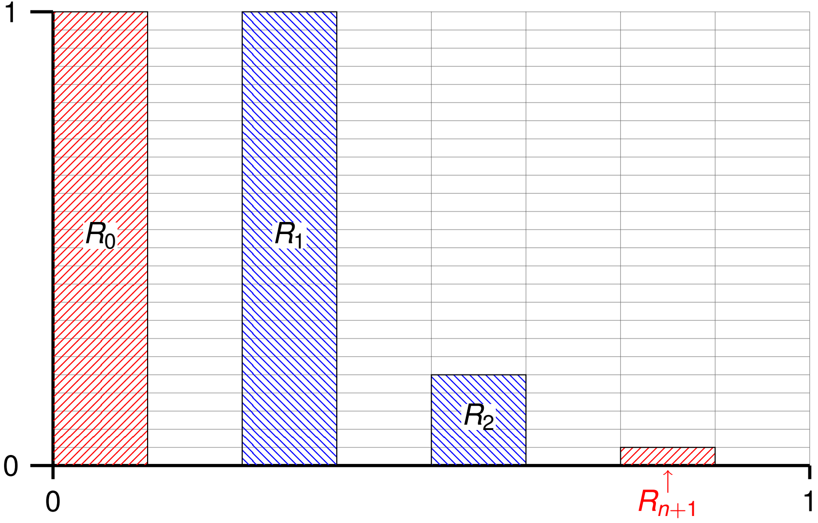
- Relaxed inital region: $\{2^{-k}\} \times [0, 1]$
  Contains “strange” points

Ok, the actual proof is slightly more complicated...



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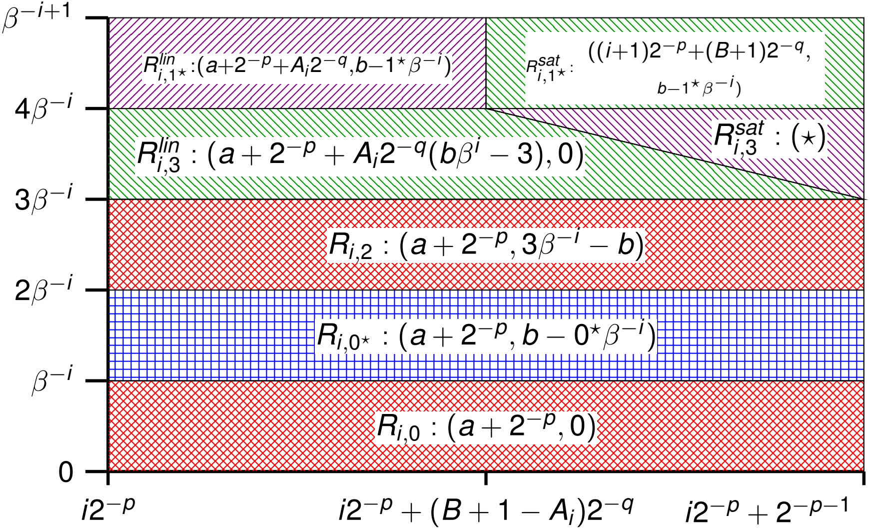

$(\star)$:$((i+1)2^{-p}+2^{-p-1}-(b\beta^i-3)(2^{-p-1}-(B+1)2^{-q}),0)$

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- Open problem for $d = 1$
- Tricky reduction for the worst case
- Tells us nothing about practical instances

Ordinary Differential Equation

Initial Value Problem

$$y(t_0) = y_0 \qquad y'(t) = f(y(t)) \qquad \forall t \in I$$

where $y : I \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^d$

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Some problems:

- Compute $y(t) \pm 2^{-n}$ for any $t \in I$ and $n \in \mathbb{N}$

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Is there a more restricted and tractable class of ODEs ?

Polynomial Initial Value Problem

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$$y(0) = 1 \qquad y'(t) = -2ty(t)^2 \quad \rightsquigarrow \quad y(t) = \frac{1}{1+t^2}$$

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Motivation:

- Simplest nontrivial class: between linear (easy) and analytic (hard)

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Motivation:

- Simplest nontrivial class: between linear (easy) and analytic (hard)
- Captures the General Purpose Analog Computer (GPAC): realistic model of computation
- Contains many interesting systems (most of Newton physics)

Quick recap on GPAC

- by Claude Shanon (1941)

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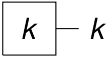
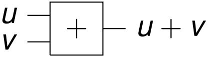
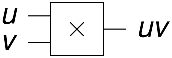


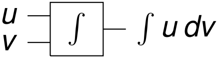
A constant unit



An adder unit



An multiplier unit



An integrator unit

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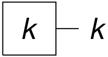
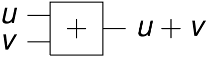
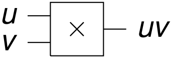


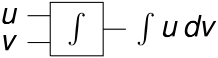
A constant unit



An adder unit



An multiplier unit



An integrator unit

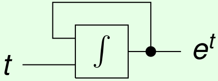
Theorem

y is generated by a GPAC iff it is a component of a PIVP

It exists !

GPAC: examples

Example (One variable, linear system)



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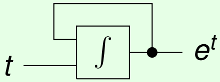

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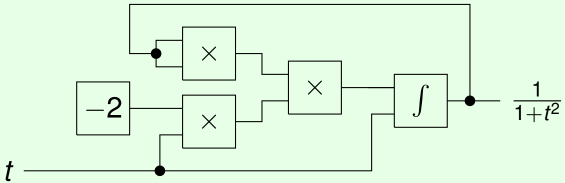

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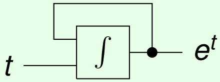

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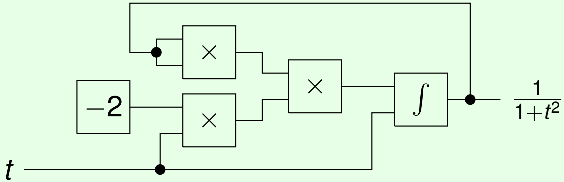

$$\begin{cases} y' = -2ty^2 \\ y(0) = 1 \\ t' = 1 \\ t(0) = 0 \end{cases}$$

Solving PIVP over unbounded domain

Assume that $y : I \to \mathbb{R}^d$ satisfies:

$$y(t_0) = y_0 \qquad y'(t) = p(y(t)) \quad \forall t \in I$$

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- Doesn't work for unbounded I
- How does the complexity depend on y_0, d, p, I ?

Solving PIVP over unbounded domain (cont.)

Theorem (Our work)

Computing $y(t) \pm 2^{-n}$ takes time:

$$\text{poly}(\deg p, \log \|y_0\|, \log \Sigma p, n, \ell(t_0, t))^d$$

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Proof details

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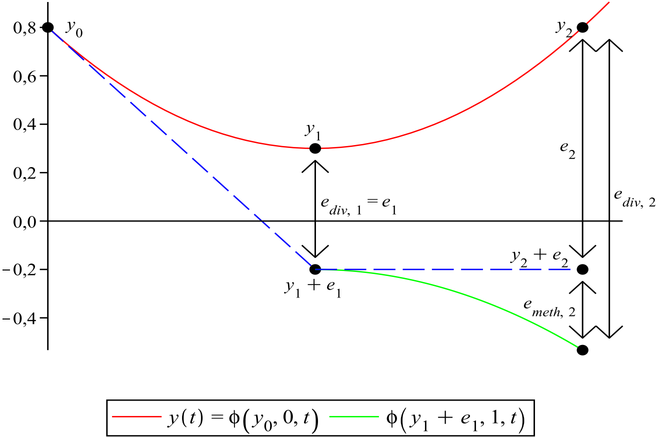

Interesting (practical ?) consequences

Compute $y(t) \pm \varepsilon$

Method	Max. Order	Number of steps
Fixed ω	$\omega - 1$	$\mathcal{O}\left(L^{\frac{\omega+1}{\omega-1}}\varepsilon^{-\frac{1}{\omega-1}}\right)$

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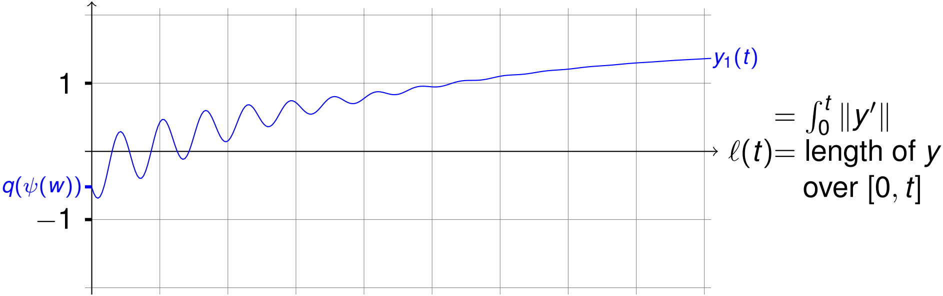

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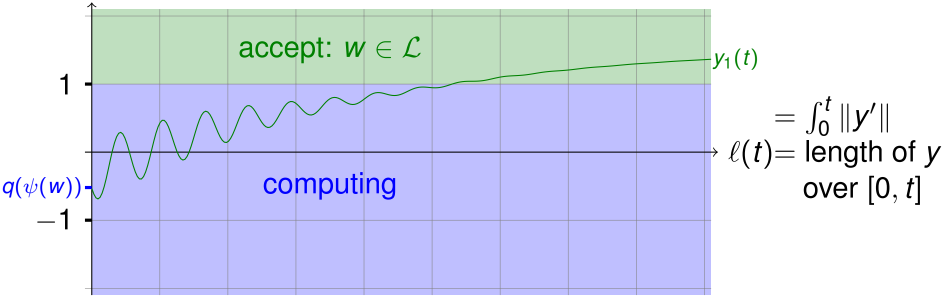

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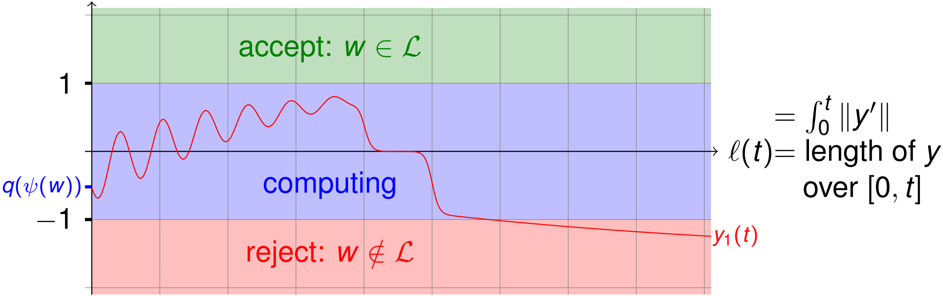

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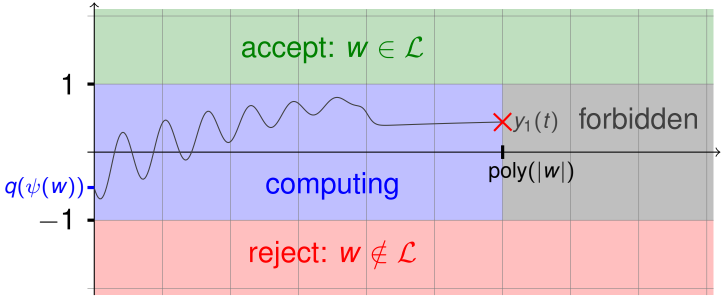

$\ell(t)$= length of y over $[0,t]$ $= \int_0^t \|y'\|$

3. if $\ell(t) \geqslant \text{poly}(|w|)$ then $|y_1(t)| \geqslant 1$

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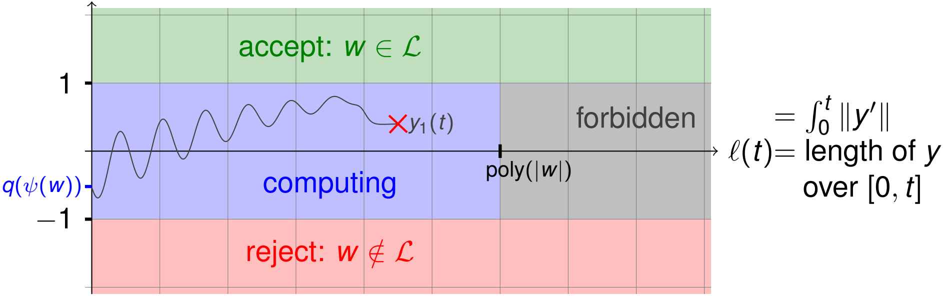

4. $\ell(t) \geqslant t$

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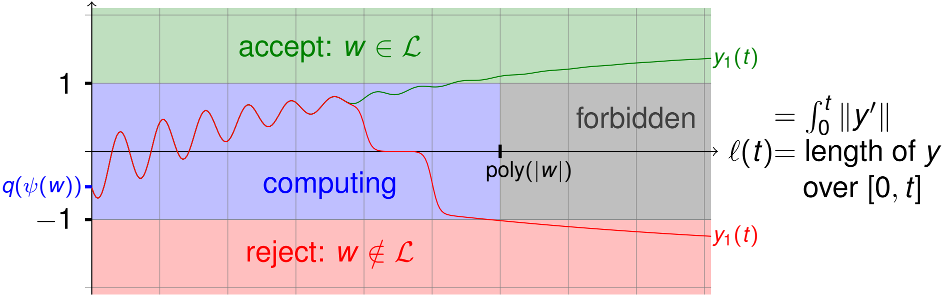

Theorem

$\mathcal{L} \in$ P if and only if $\mathcal{L}$ is polytime-recognizable.

PIVP hardness

Corollary

The following problem is *P*-complete:
Input: $y_0 \in \mathbb{Q}^d$, $\Upsilon \in \mathbb{N}$ in unary, p polynomial

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Decide: $y(1) \geqslant 1$?

Proof gem: iteration with differential equations

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Iterate $\tilde{f}$ with a PIVP: $y(n) \approx \tilde{f}^{[n]}([x])$

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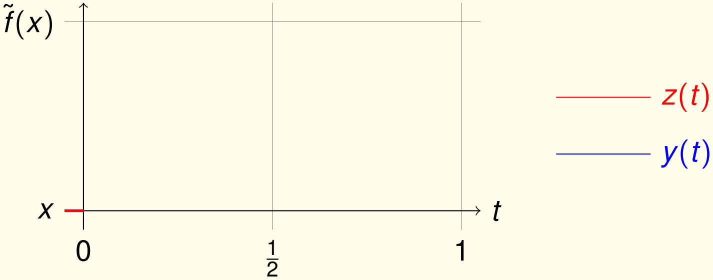

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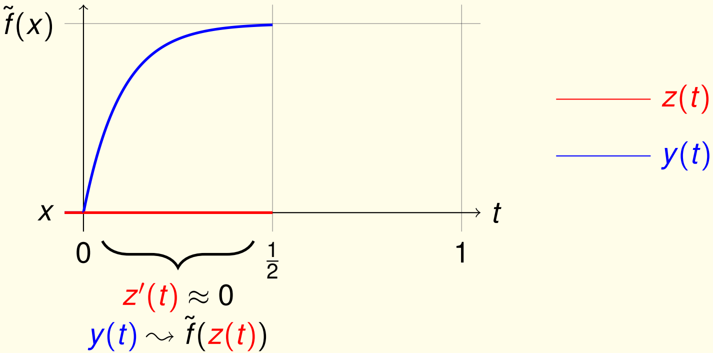

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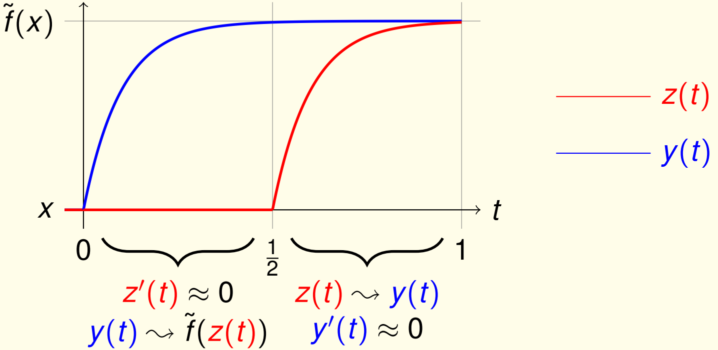

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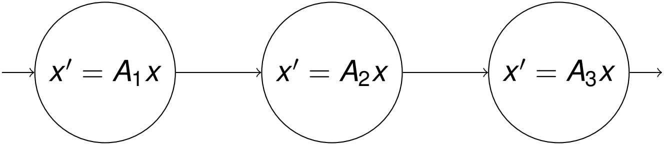
Future work:

- Extend PIVP solving to more general ODEs
- More general reachability
- Study practical complexity of PIVP solving
- Other measures of complexity
- More efficient algorithm for systems with more properties ?

Linear Hybrid Automata



- Finite number of control states
- Each state has a linear continuous dynamic: $x' = Ax$
- Nondeterministic transitions between states (no guards)

Small math recap

The solution to a linear system of differential equations

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The solution to a linear system of differential equations

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is given by:

$$x(t) = x_0 e^{At}$$

where the exponential of matrices is given by

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

Problems

Reachability:

- Point to point (Orbit problem)
- Point to region (Hitting problem)
- Escape problem

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Transformation:

- What kind of transformations can be achieved ?

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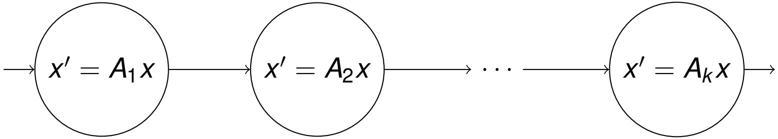
Given *algebraic* matrices $A_1, \ldots, A_k, C$.



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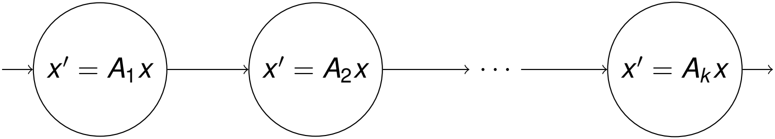

Definition (Matrix-Exponential Problem)

Decide if there exists $t_1, \ldots, t_k \geqslant 0$ such that:

$$\prod_{i=1}^{k} e^{A_i t_i} = C.$$

Matrix-Exponential Semigroup Prolem

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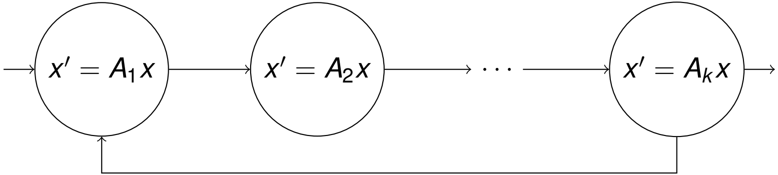

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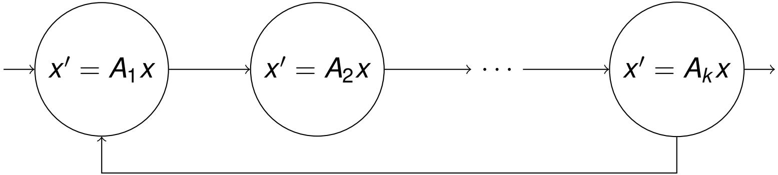

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Definition (Matrix-Exponential Semigroup Problem)

Decide if C belongs to the semigroup generated by:

$$\{\exp(A_i t) : t \geqslant 0, i = 1, \ldots, k\}.$$

Results:

Theorem (Commutative case)

If the matrices $A_1, \ldots, A_k$ commute, Matrix-Exponential and Matrix-Exponential Semigroup problems are equivalent and decidable.

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Theorem (General case)

In general, Matrix-Exponential and Matrix-Exponential Semigroup problems are undecidable.

Questions ?

- Do you have any questions ?