Continuous models of computation: computability, complexity, universality

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Digital vs analog computers
Digital vs analog computers
Church Thesis

All reasonable models of computation are equivalent.
Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.
Polynomial Differential Equations

\[ \begin{align*}
    &k \\
    &\frac{u}{v} \quad \times \\
    &\int \\
    &u + v
\end{align*} \]

General Purpose Analog Computer

Differential Analyzer

Newton mechanics

Reaction networks:
- chemical
- enzymatic

polynomial differential equations:
\[
\begin{align*}
    y(0) &= y_0 \\
    y'(t) &= p(y(t))
\end{align*}
\]

- Rich class
- Stable (+,×,○,/,,ED)
- No closed-form solution
Example of dynamical system

\[ \ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \]

\[
\begin{cases}
  y_1' = y_2 \\
  y_2' = -\frac{g}{\ell} y_3 \\
  y_3' = y_2 y_4 \\
  y_4' = -y_2 y_3
\end{cases}
\]

\[
\begin{cases}
  y_1 = \theta \\
  y_2 = \dot{\theta} \\
  y_3 = \sin(\theta) \\
  y_4 = \cos(\theta)
\end{cases}
\]
Computing with the GPAC

Generable functions

\[
\begin{align*}
&\begin{aligned}
y(0) &= y_0 \\
y'(x) &= p(y(x))
\end{aligned} & x \in \mathbb{R}
\end{align*}
\]

\[ f(x) = y_1(x) \]

Shannon’s notion

\[
\sin, \cos, \exp, \log, \ldots
\]

Strictly weaker than Turing machines [Shannon, 1941]

\[
\begin{align*}
f(x) &= \lim_{t \to \infty} y_1(t) \\
x \in \mathbb{R}
\end{align*}
\]

Modern notion

\[
\sin, \cos, \exp, \log, \Gamma, \zeta, \ldots
\]

Turing powerful [Bournez et al., 2007]
Computing with the GPAC

Generable functions

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\begin{cases}
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\begin{aligned}
    y(0) &= q(x) \\
    y'(t) &= p(y(t))
\end{aligned}
\quad x \in \mathbb{R} \\
\quad t \in \mathbb{R}_+
\]

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Turing powerful

[Bournez et al., 2007]
From discrete to real computability

Computable Analysis: lift Turing computability to real numbers

[Ko, 1991; Weihrauch, 2000]

Definition

$x \in \mathbb{R}$ is computable iff there exists a computable $f : \mathbb{N} \to \mathbb{Q}$ such that:

$$|x - f(n)| \leq 10^{-n}$$

$n \in \mathbb{N}$

Examples: rational numbers, $\pi, e, ...$

Beware: there exists uncomputable real numbers!
From discrete to real computability

Computable Analysis: lift Turing computability to real numbers

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| $n$ | $f(n)$ | $|\pi - f(n)|$ |
|-----|--------|---------------|
| 0   | 3      | $0.14 \leq 10^{-0}$ |
| 1   | 3.1    | $0.04 \leq 10^{-1}$ |
| 2   | 3.14   | $0.001 \leq 10^{-2}$ |
| 10  | 3.1415926535 | $0.9 \cdot 10^{-10} \leq 10^{-10}$ |
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Beware: there exists uncomputable real numbers!
From discrete to real computability

**Definition (Computable function)**

$f : [a, b] \rightarrow \mathbb{R}$ is computable iff

\[
| x - y | \leq 10^{-m(n)} \Rightarrow | f(x) - f(y) | \leq 10^{-n(x,y)} \quad \forall x, y \in \mathbb{R}, n \in \mathbb{N}
\]

**Polytime complexity**

Add “polynomial time computable” everywhere.
Definition (Computable function)

$f : [a, b] \to \mathbb{R}$ is computable iff $\exists \ m : \mathbb{N} \to \mathbb{N}$, computable functions such that:

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$m$ : modulus of continuity
**Definition (Computable function)**

$f : [a, b] \to \mathbb{R}$ is computable iff \( \exists m : \mathbb{N} \to \mathbb{N}, \psi : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \)
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|f(r) - \psi(r, n)| \leq 10^{-n} \quad r \in \mathbb{Q}, n \in \mathbb{N}
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Examples: polynomials, \( \sin, \exp, \sqrt{\cdot} \)

Note: all computable functions are continuous

Beware: there exists (continuous) uncomputable real functions!
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**Polytime complexity**

Add “polynomial time computable” everywhere.
Equivalence with computable analysis

Definition (Bournez et al)

\( f \) computable by GPAC if \( \exists p \) polynomial such that \( \forall x \)

\[
y(0) = (x, 0, \ldots, 0) \quad y'(t) = p(y(t))
\]

satisfies \( |f(x) - y_1(t)| \leq y_2(t) \) et \( y_2(t) \xrightarrow{t \to \infty} 0. \)
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**Theorem (Bournez, Campagnolo, Graça, Hainry)**

\( f : [a, b] \to \mathbb{R} \) computable \( \iff \) \( f \) computable by GPAC

\[ y_1(t) \xrightarrow{t \to \infty} f(x) \quad y_2(t) = \text{error bound} \]
Complexity of analog systems

- Turing machines: $T(x) = \text{number of steps to compute on } x$
Complexity of analog systems

- **Turing machines**: $T(x) =$ number of steps to compute on $x$
- **GPAC**: time contraction problem

**Tentative definition**

$T(x, \mu) =$ first time $t$ so that $|y_1(t) - f(x)| \leq e^{-\mu}$

$y(0) = (x, 0, \ldots, 0) \quad y' = p(y)$

![Graph showing $y_1(t)$ and $f(x)$ over time $t$.]
Complexity of analog systems

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Graphs showing $x$, $y_1(t)$, $f(x)$, $z_1(t)$, and $f(x)$ on $t$ axis.
Complexity of analog systems

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- **GPAC**: time contraction problem

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\[
w(t) = y(e^{e^t})
\]
Complexity of analog systems

- **Turing machines**: $T(x) =$ number of steps to compute on $x$
- **GPAC**: time contraction problem → open problem

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$z(t) = y(e^{e^t})$

$w(t) = y(e^{e^{et}})$

**Problem**

All functions have constant time complexity.
Time-space correlation of the GPAC

\[ y(0) = q(x) \quad y' = p(y) \]

\[ z(t) = y(e^t) \]

Observation: Time scaling costs "space". Time complexity for the GPAC must involve time and space!
Time-space correlation of the GPAC

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Extra component: \( w(t) = e^t \)
Time-space correlation of the GPAC

\[ y(0) = q(x) \quad y' = p(y) \]

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**Observation**

Time scaling costs “space”.

Time complexity for the GPAC must involve time and space!
Complexity of solving polynomial ODEs

\[ y(0) = x \quad y'(t) = p(y(t)) \]

**Theorem (Graça, Pouly) [TCS 2016]**

If \( y(t) \) exists, one can compute \( p, q \) such that

\[ \left| \frac{p}{q} - y(t) \right| \leq 2^{-n} \]

in time

\[ \text{poly} \left( \text{size of } x \text{ and } p, n, \ell(t) \right) \]

where \( \ell(t) = \int_0^t \max(1, \|y(u)\|^\deg(p)) \, du \approx \text{length of the curve} \)

length of the curve = complexity = ressource
Characterization of polynomial time

**Definition:** $L \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial, } \forall \text{ word } w$

$$y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}$$

$\ell(t) = \text{length of } y$
Characterization of polynomial time

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\]

- **accept**: \( w \in \mathcal{L} \)
- **computing**

satisfies

1. if \( y_1(t) \geq 1 \) then \( w \in \mathcal{L} \)
**Definition**: \( \mathcal{L} \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial, } \forall \text{ word } w \)

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\]

- **accept**: \( w \in \mathcal{L} \)
- **reject**: \( w \notin \mathcal{L} \)

satisfies

1. if \( y_1(t) \leq -1 \) then \( w \notin \mathcal{L} \)
**Characterization of polynomial time**

**Definition**: \( \mathcal{L} \in \text{ANALOG-PTIME} \iff \exists p \text{ polynomial}, \forall \text{ word } w \)

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y(0) = (\psi(w), |w|, 0, \ldots, 0) \quad y' = p(y) \quad \psi(w) = \sum_{i=1}^{|w|} w_i 2^{-i}
\]

\(y(0)\) represents the initial state of the computation, where \(\psi(w)\) is a function of the word \(w\), and \(|w|\) is the length of the word. The function \(\psi(w)\) computes the sum of \(w_i 2^{-i}\) for each \(w_i\) in \(w\).

The diagram illustrates the state transition of the computation, with states labeled as 'accept', 'computing', and 'reject'. The condition for acceptance is met if \(\ell(t) \geq \text{poly}(|w|)\) and \(|y_1(t)| \geq 1\).
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Theorem (JoC 2016; ICALP 2016)

\[ \text{PTIME} = \text{ANALOG-PTIME} \]
characterization of real polynomial time

**Definition:** \( f : [a, b] \rightarrow \mathbb{R} \) in ANALOG-\( P_{\mathbb{R}} \) \iff \exists p \) polynomial, \( \forall x \in [a, b] \\
y(0) = (x, 0, \ldots, 0) \quad y' = p(y) \)
Characterization of real polynomial time

**Definition**: \( f : [a, b] \rightarrow \mathbb{R} \) in \( \text{ANALOG-P}_\mathbb{R} \iff \exists p \text{ polynomial}, \forall x \in [a, b] \)

\[
y(0) = (x, 0, \ldots, 0) \quad y' = p(y)
\]
satisfies:

1. \(|y_1(t) - f(x)| \leq 2^{-\ell(t)}\)
   «greater length \(\Rightarrow\) greater precision»

2. \(\ell(t) \geq t\)
   «length increases with time»

Theorem [JoC 2016; ICALP 2016]

\( f : [a, b] \rightarrow \mathbb{R} \) computable in polynomial time \(\iff f \in \text{ANALOG-P}_\mathbb{R} \).
Characterization of real polynomial time

**Definition**: \( f : [a, b] \rightarrow \mathbb{R} \) in \( \text{ANALOG-P}_{\mathbb{R}} \) \( \iff \) \( \exists p \) polynomial, \( \forall x \in [a, b] \)
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**Theorem** [JoC 2016; ICALP 2016]
\( f : [a, b] \rightarrow \mathbb{R} \) computable in polynomial time \( \iff f \in \text{ANALOG-P}_{\mathbb{R}}. \)
Theorem [JoC 2016; ICALP 2016]

- $\mathcal{L} \in \text{PTIME}$ if and only if $\mathcal{L} \in \text{ANALOG-PTIME}$
- $f : [a, b] \rightarrow \mathbb{R}$ computable in polynomial time $\iff f \in \text{ANALOG-P}_R$

- Analog complexity theory based on length
- time of Turing machine $\iff$ length of the GPAC
- Purely continuous characterization of PTIME
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
Universal differential equations

Generable functions

subclass of analytic functions

Computable functions

any computable function
**Theorem (Rubel)**

There exists a **fixed** polynomial \( p \) and \( k \in \mathbb{N} \) such that for any continuous functions \( f \) and \( \varepsilon \), there exists a solution \( y \) to

\[
p(y, y', \ldots, y^{(k)}) = 0
\]

such that

\[
|y(t) - f(t)| \leq \varepsilon(t).
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Theorem (Rubel)

There exists a **fixed** polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists a solution $y$ to

$$p(y, y', \ldots, y^{(k)}) = 0$$

such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$

**Problem**: Rubel is «cheating». 
Theorem

There exists a **fixed** polynomial $p$ and $k \in \mathbb{N}$ such that for any continuous functions $f$ and $\varepsilon$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, \ y'(0) = \alpha_1, \ldots, \ y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** and this solution satisfies such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$
Future work

Reaction networks:
- chemical
- enzymatic

\[ y' = p(y) \]
\[ y' = p(y) + e(t) \]

- Finer time complexity (linear)
- Nondeterminism
- Robustness
- « space» complexity
- Other models
- Stochastic