Linear Dynamical Systems

Control Theory

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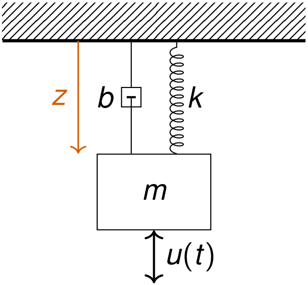
Example : mass-spring-damper system



Model with external input $u(t)$

State : $X = z \in \mathbb{R}$

Equation of motion :

$$mz'' = -kz - bz' + mg + u$$

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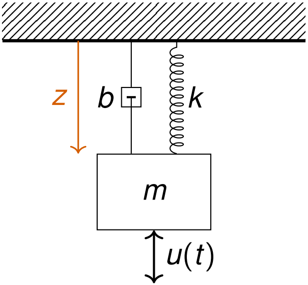

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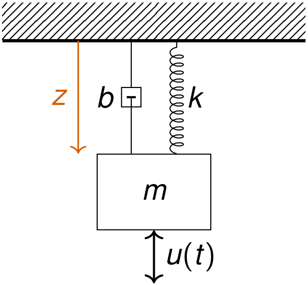

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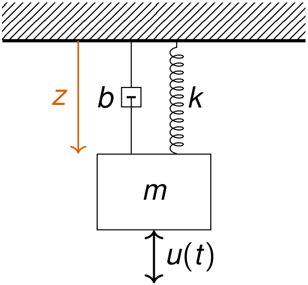

Model with external input $u(t)$

$\rightarrow$ Linear time invariant system

$$X' = AX + Bu$$

with some constraints on u.

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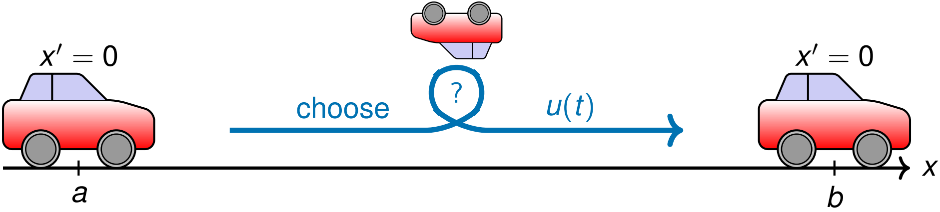

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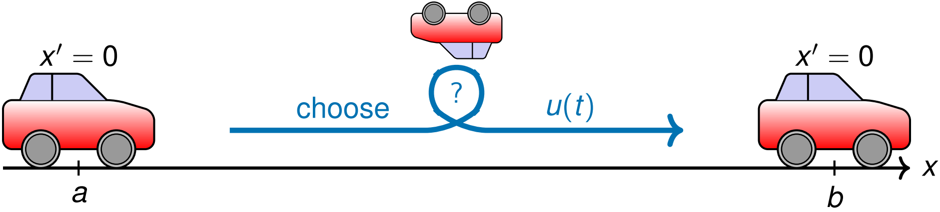
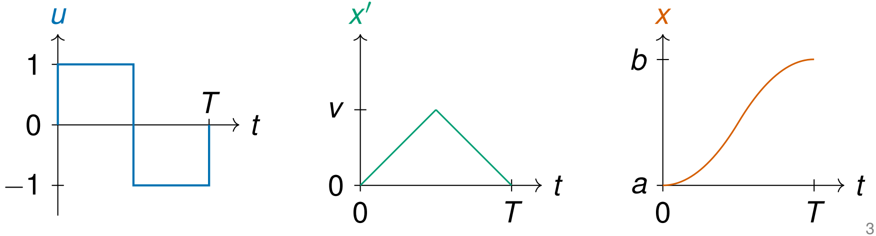


Possible solution :



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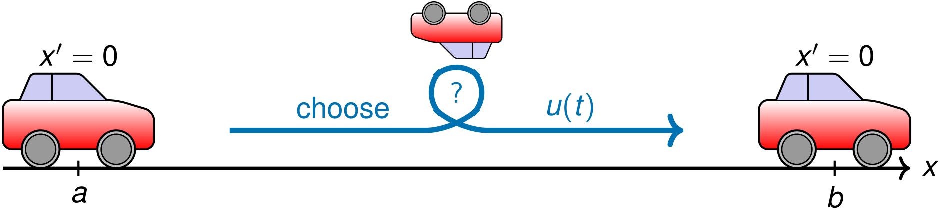
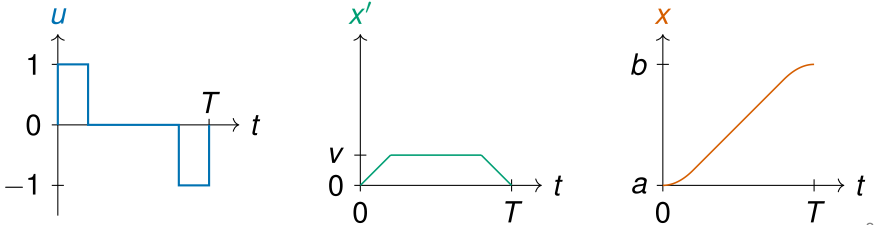


More realistic solution :



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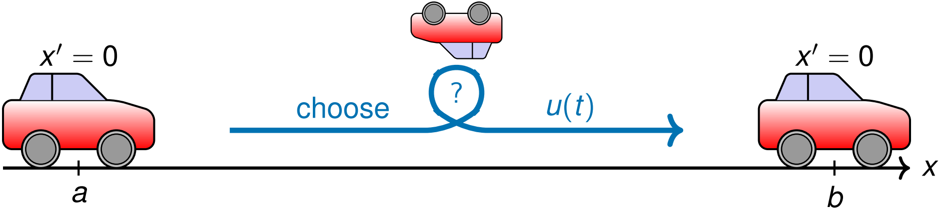

Rephrasing the problem :

$$\begin{cases} x' = y \\ y' = u \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \Leftrightarrow X' = AX + U$$

Starting from $(x, y) = (a, 0)$, try to reach $(x, y) = (b, 0)$.

This is a point-to-point reachability problem.

The problem

LTI Reachability problem

- a source $y \in \mathbb{Q}^n$,
- a target $z \in \mathbb{Q}^n$,
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- a set of controls $U \subseteq \mathbb{R}^n$,

decide if $\exists T \geqslant 0$, $u : [0, T] \to U$ *measurable* such that $x(T) = z$ where

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Warning : u **does not need to be "describable"**, *e.g.* piecewise polynomial. Otherwise, completely changes the nature of the problem.

Bigger picture

Continuous Reachability problem

- a source $y \in \mathbb{Q}^n$,
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Generally undecidable :

- for nonlinear systems, even without control ($U = \{0\}$)
- piecewise constant derivative systems (PCD), still *no control*
- linear saturated systems (at least for discrete systems), no control

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But do they really ?

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But also :

- assumptions on A (typically spectral)
- assumptions on U
- restrictions on acceptable u

Two known extreme cases

- When we have no control :

$$U = \{0\} \quad \text{and} \quad x'(t) = Ax + u(t) \qquad \Leftrightarrow \qquad x(t) = e^{At}x(0).$$

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Theorem (Folklore)

Given $y, z \in \mathbb{Q}^n$ and $A \in \mathbb{Q}^{n\times n}$, $B \in \mathbb{Q}^{n\times m}$, it is decidable whether $\exists T \geqslant 0$ and $u : [0, T] \to B\mathbb{R}^m$ measurable such that $x(0) = y$ and $x(T) = z$ where

$$x'(t) = Ax(t) + u(t)$$

Back to the future

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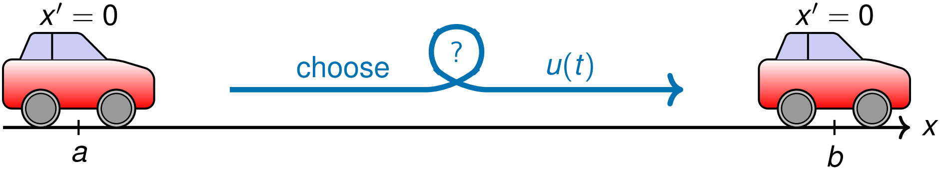

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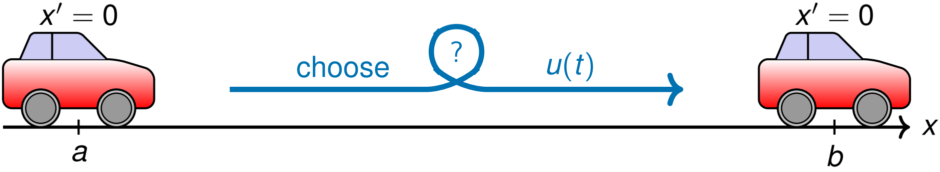

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Very few decidability results in the literature in this case.

Our results : decidability

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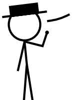

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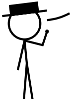

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Are you sure you cannot do better ?

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A deep conjecture in transcendental number theory. Widely believed to be true and totally open.

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and Schanuel's conjecture is true.

Theorem (Wilkie and MacIntyre)

If Schanuel's conjecture is true, then, for each $k \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos\restriction_{[0,k]}, \sin\restriction_{[0,k]})$ is decidable.

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This is a well-known “hard” problem.

Hardness (cont.)

Taking $U = \{0\}$ is cheating :

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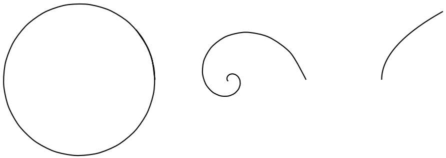

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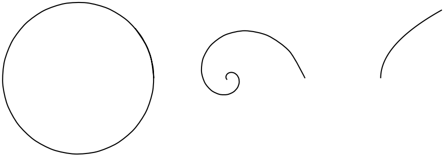
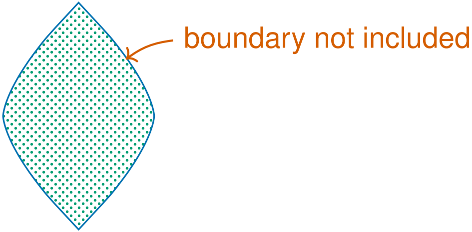


- when $U = B[-1, 1]^m$, reachable set is open



This is completely different !

Our results : hardness

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Continuous Nontangential Skolem problem

Given a matrix $A \in \mathbb{Q}^{n\times n}$ and $c, x_0 \in \mathbb{Q}^n$, decide if $\exists T \geqslant 0$ such that $f(t) = 0$ and $f'(t) \neq 0$ where $f(t) = c^T e^{At} x_0 = 0$.

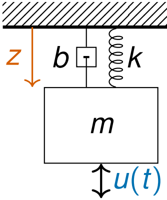
It is essentially as hard as the Continuous Skolem problem.

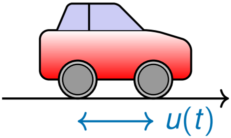
Conclusion (continuous case)

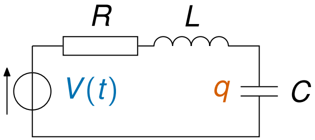
LTI reachability problem : find T and u such that

$$x(0) = 0, \qquad x'(t) = Ax(t) + Bu(t), \qquad u(t) \in [-1, 1]^m$$

satisfies $x(T) =$ target. Very natural problem in control theory.







Point reachability is

- decidable in dimension 2 or with spectral constraints,
- conditionally decidable with real eigenvalues,
- conditionally decidable in bounded time,

Set reachability is Nontangential Continuous Skolem hard.

The continuous case is much harder than expected. What about the discrete case ?

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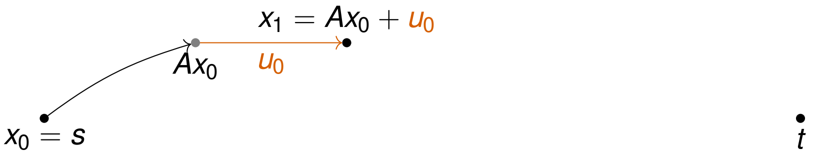

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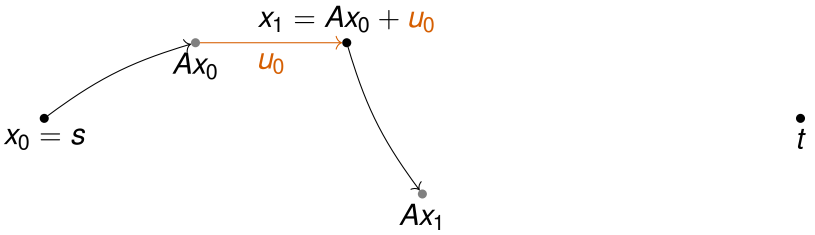

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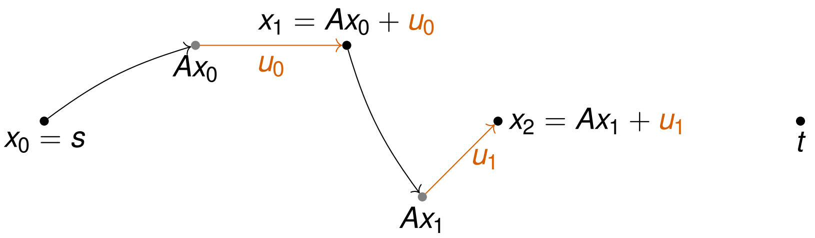

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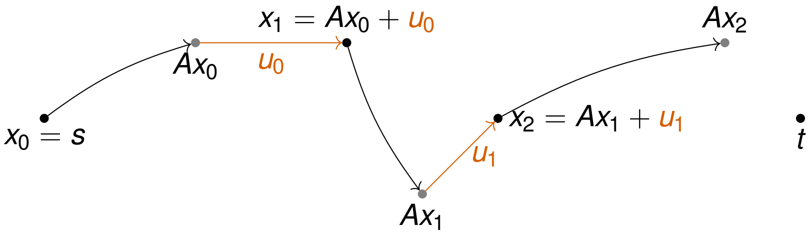

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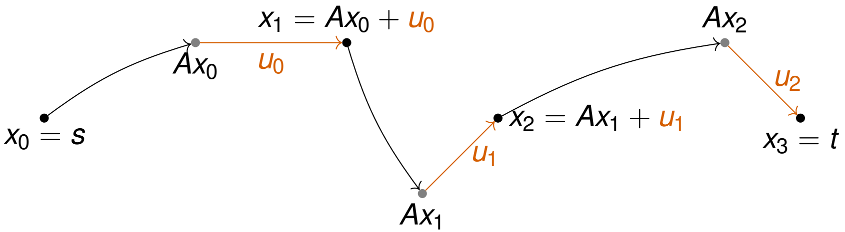

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Almost no exact results for other classes of U in particular when U is bounded (which is the most natural case).

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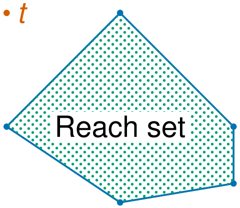

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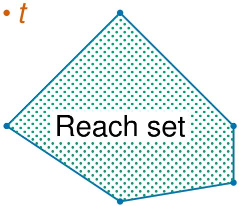

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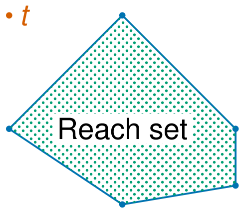

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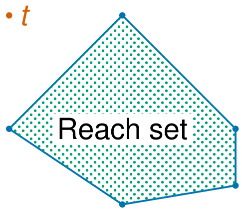

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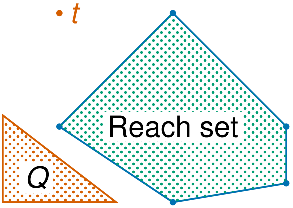
Remark : in fact we can decide reachability to a convex polytope Q.



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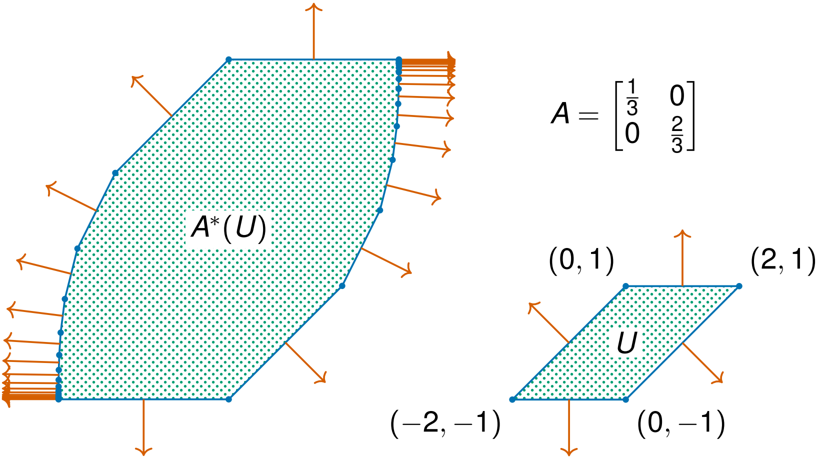
Why is this problem hard

The reachable set $A^*(U)$ can have **infinitely** many faces.



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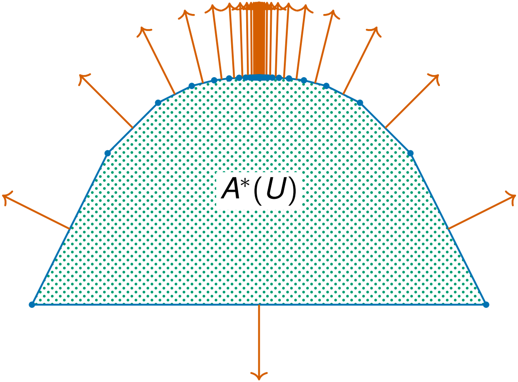
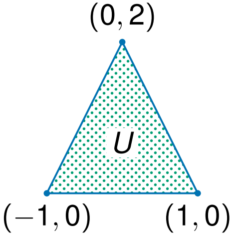
The reachable set $A^*(U)$ can have **faces of lower dimension** : the "top" extreme point does not belong to any facet.



$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$



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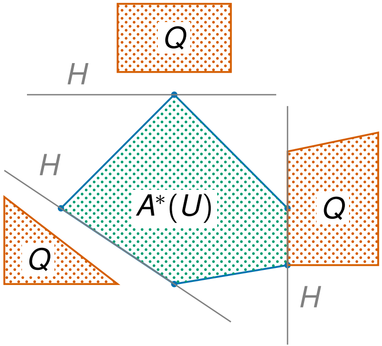

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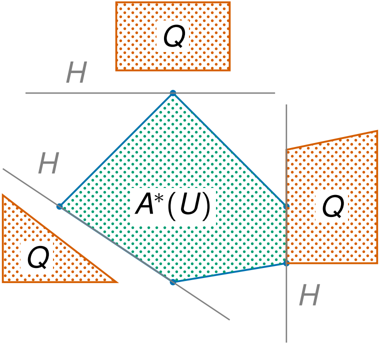

Further difficulty : a separating hyperplane may not be supported by a facet of either $A^*(U)$ or Q.

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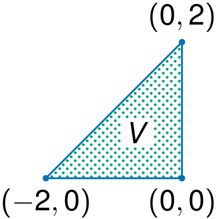

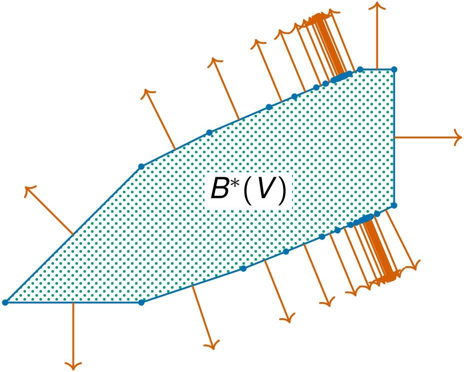

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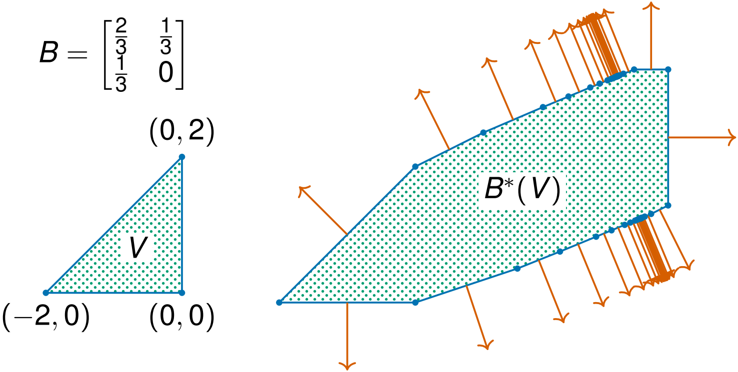

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Theorem (Non-reachable instances)

There is a separating hyperplane with algebraic coefficients.

Conclusion on control

Exact reachability for LTI systems :

- decidability crucially depends on the shape of the control set
- even with convex bounded inputs, the problem is very hard (Skolem/Positivity, open for 70 years)
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Despite an extensive literature in control theory, the decidability control problems is still very open.