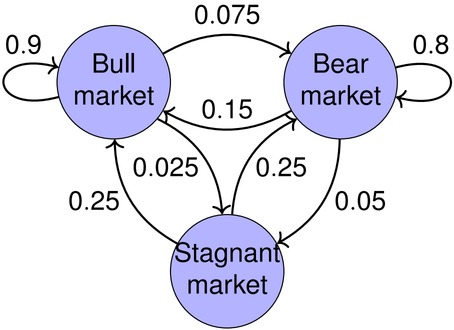
Linear Dynamical Systems

Overview

Amaury Pouly

Examples: while loop, Markov chain



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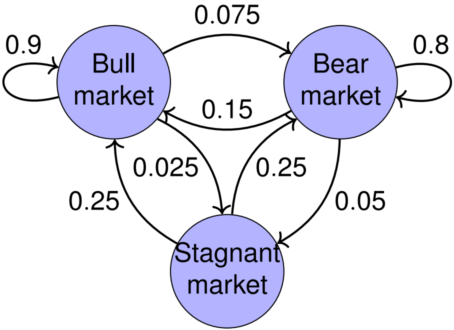

State: $X = (p_{bull}, p_{bear}, p_{stag}) \in [0,1]^3$
Transitions:

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$

$\rightarrow$ Linear dynamical system

$$X_{n+1} = AX_n$$

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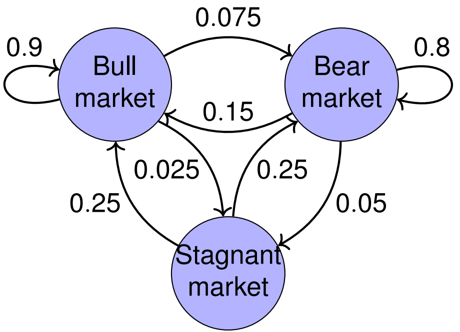

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Linear loop

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while $p_{bull} \leqslant 1/2$ do

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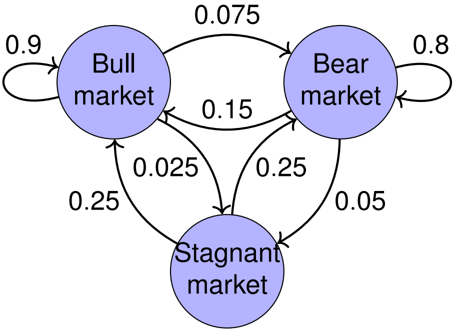

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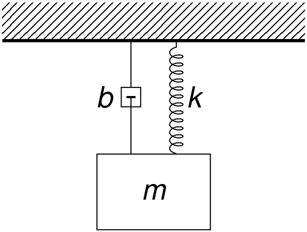
The loop terminates if and only if the probability of a bull market is $> 1/2$.

Example: mass-spring-damper system



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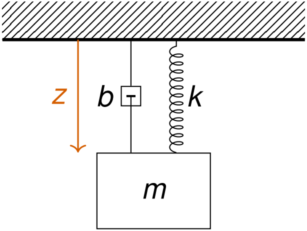

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Equation of motion:

$$mz'' = -kz - bz' + mg$$

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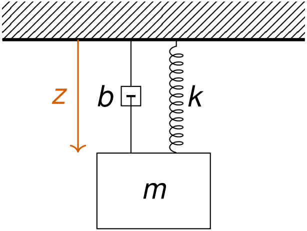

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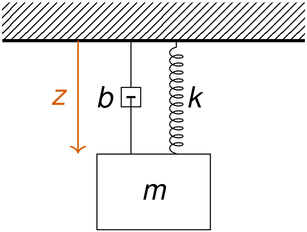

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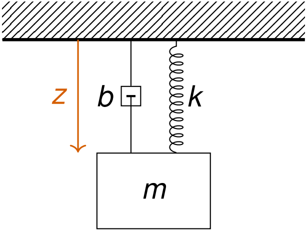

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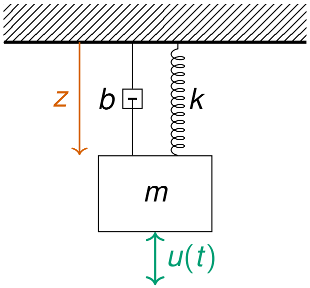

with external input $u(t)$.

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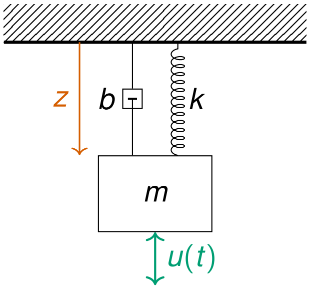

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$$X' = AX + Bu$$

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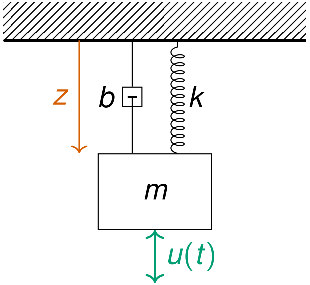

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Can be used to model a car suspension.

Linear dynamical systems

Discrete case

$$x(n+1) = Ax(n)$$

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
- ....

Continuous case

$$x'(t) = Ax(t)$$

- biology,
- physics,
- probabilistic model checking,
- electrical circuits,
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Typical questions

- reachability
- safety

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Typical questions

- reachability
- safety
- controllability
- optimal control
- feedback control
- ...

More complicated programs

Linear loop with if

$x := 2^{-10}$
$y := 1$
while $y \geqslant x$ do
    if $y \geqslant 2x$ then
$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
    else
$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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- invariant* synthesis also hard

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$\rightsquigarrow$

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Overapproximate behaviours

- reachability still undecidable
- invariant synthesis possible

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Does this program halt?

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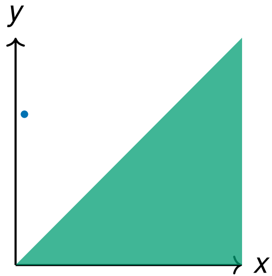

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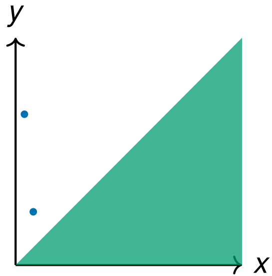

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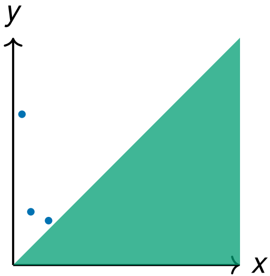

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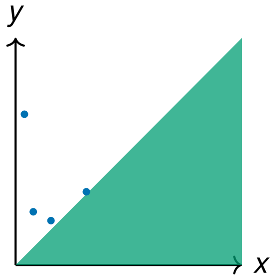

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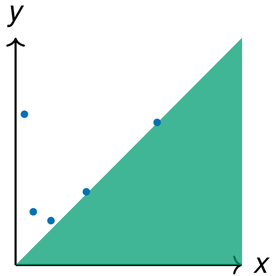

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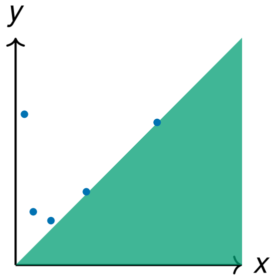

Certificate of non-termination:

$$x^2 y - x^3 = \frac{1023}{1073741824} \tag{1}$$

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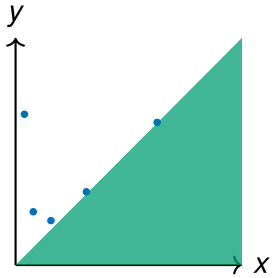

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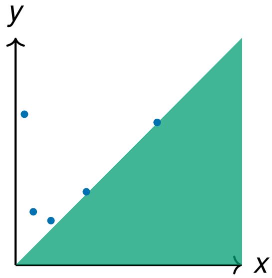
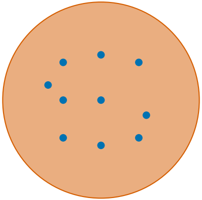


- (1) is an invariant: it holds at every step
- (1) implies the guard is true

Invariants

invariant = overapproximation of the reachable states



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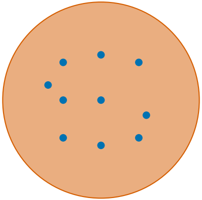
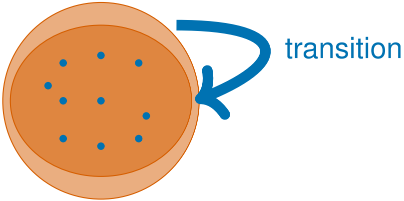
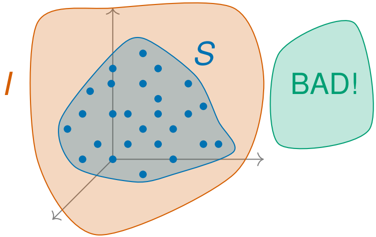


inductive invariant = invariant preserved by the transition relation



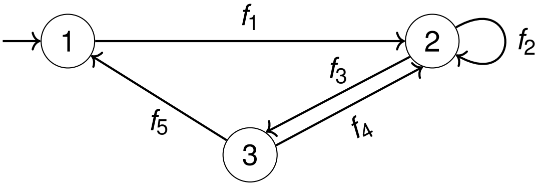
Why Invariants?



*The classical approach to the verification of temporal safety properties of programs requires the construction of **inductive invariants** [...]. **Automation of this construction is the main challenge in program verification.***

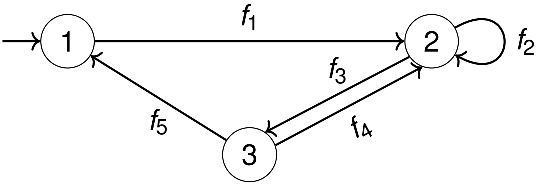
D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
Invariant Synthesis for Combined Theories, 2007

Affine programs



Affine programs

- Nondeterministic branching (no guards)



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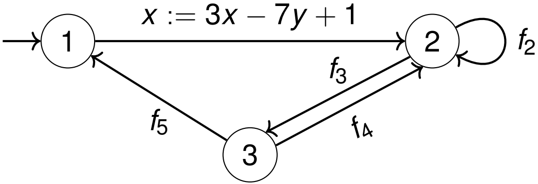

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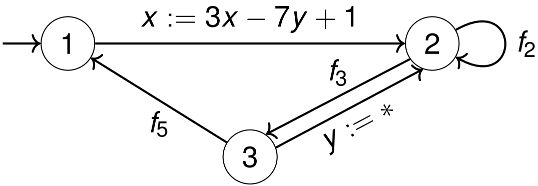
- Allow nondeterministic assignments ($x := *$)



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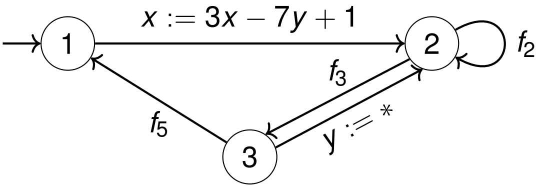
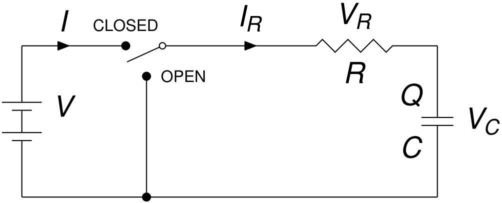
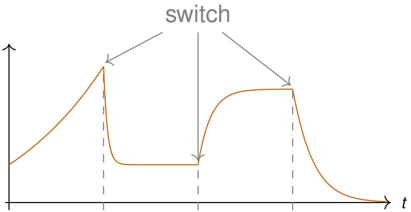


- Can overapproximate complex programs
- Covers existing formalisms:
  finite, probabilistic, quantum, quantitative automata

RC circuit





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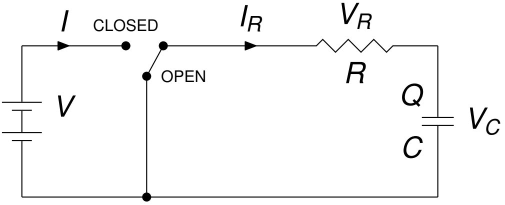

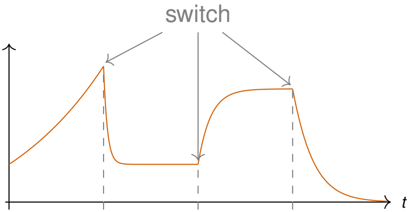

OPEN

$$\boxed{\begin{aligned} \dot{I} &= 0 \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}}$$

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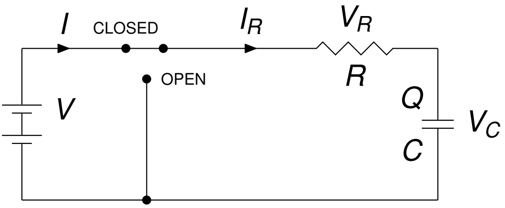

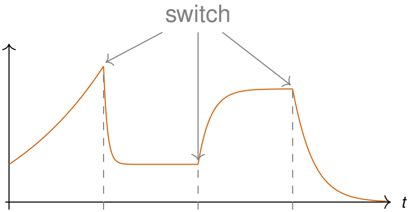

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CLOSED

$$
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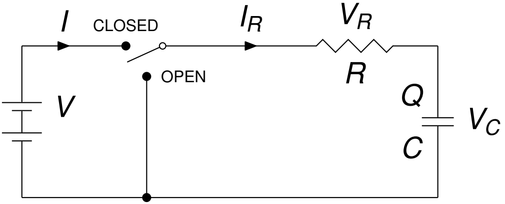
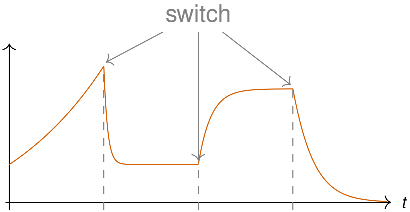
$$

RC circuit





OPEN

$$\begin{aligned} \dot{I} &= 0 \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}$$

OPEN → CLOSED:

$$\begin{aligned} I &:= \frac{1}{R}(V - V_C) \\ I_R &:= \frac{1}{R}(V - V_C) \\ V_R &:= V - V_C \end{aligned}$$

CLOSED → OPEN:

$$\begin{aligned} I &:= 0 \\ I_R &:= -\frac{1}{R} V_C \\ V_R &:= -V_C \end{aligned}$$

CLOSED

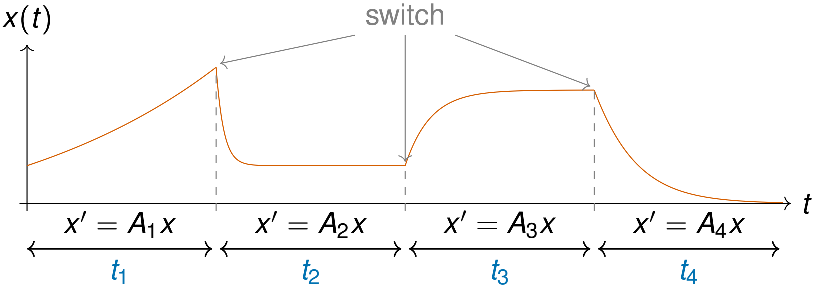
$$\begin{aligned} \dot{I} &= -\frac{1}{RC} I_R \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}$$

Switching systems

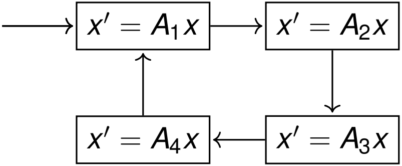
$$\begin{tikzcd} {} \arrow[r] & \boxed{x' = A_1 x} \arrow[r] & \boxed{x' = A_2 x} \arrow[d] \\ & \boxed{x' = A_4 x} \arrow[u] & \boxed{x' = A_3 x} \arrow[l] \end{tikzcd}$$

Restricted hybrid system:

- linear dynamics
- no guards (nondeterministic)
- no discrete updates



Switching systems



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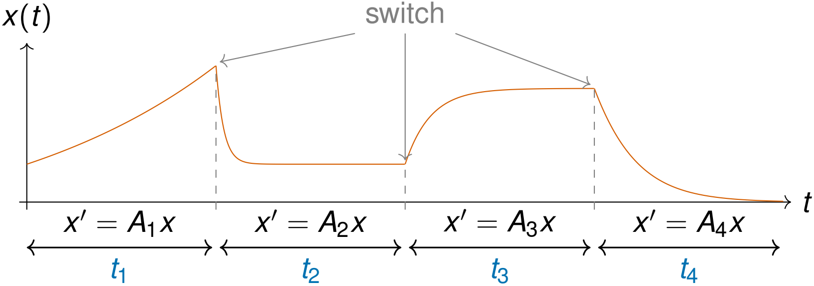
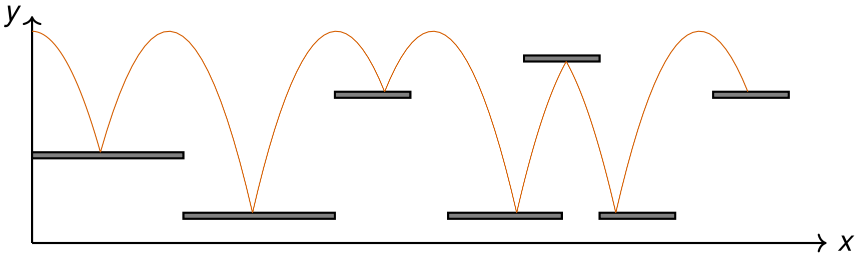


- reachability also undecidable
- invariant synthesis possible

Going hybrid: a bouncing ball



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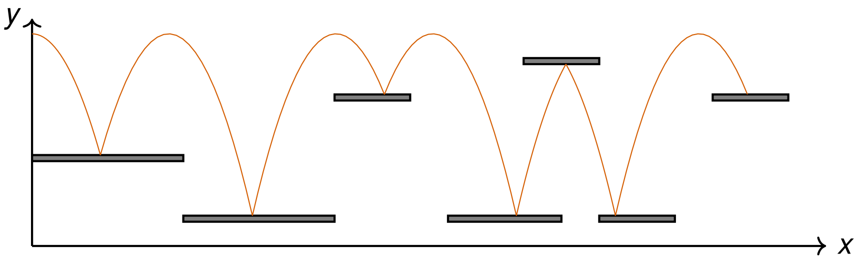


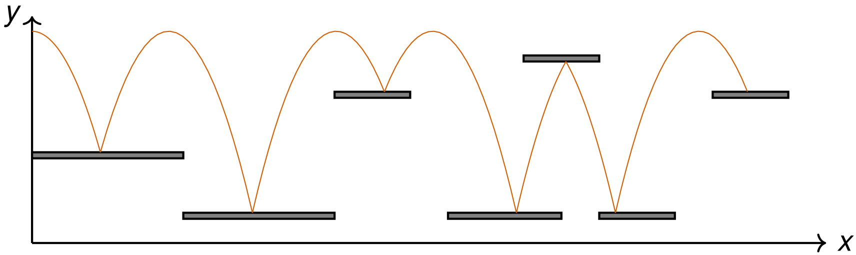
$v_y := -v_y$

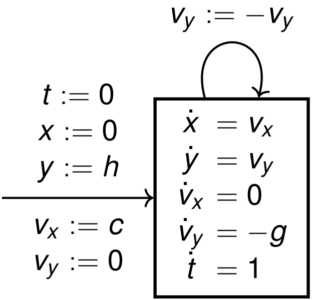
$t := 0$
$x := 0$
$y := h$
$v_x := c$
$v_y := 0$

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= 0 \\ \dot{v}_y &= -g \\ \dot{t} &= 1 \end{aligned}$$

- ▶ affine program: collision
- $+$ linear differential equation: mechanics
- $=$ linear hybrid automaton

Going hybrid: a bouncing ball





- affine program: collision
- \+ linear differential equation: mechanics
- = linear hybrid automaton

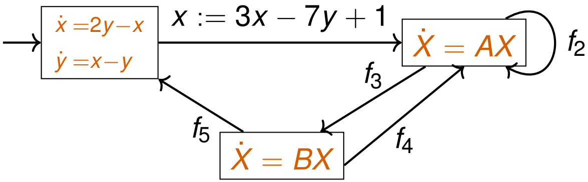
Invariants:

- $v_x = c$
- $x = tc$
- $v_y^2 + 2g(y - h) = 0$

recover conservation of energy!

Linear Hybrid Automata

- Nondeterministic branching (no guards)
- All assignments are affine
- Linear differential equations in each location



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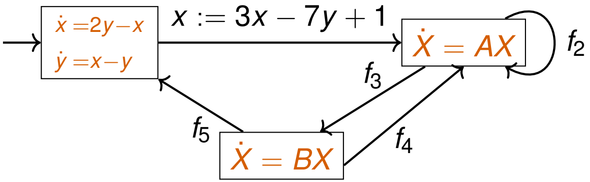
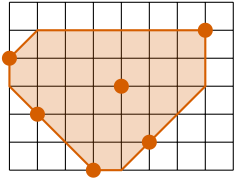


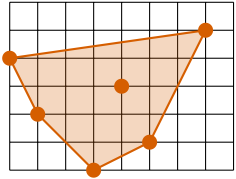
- More general than affine programs
- More general than linear differential equations

Which invariants?



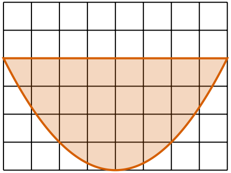
Octagons

$\leqslant$



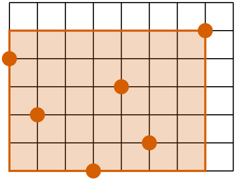
Polyhedrons

$\leqslant$

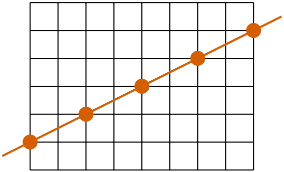


Semialgebraic sets

$\leqslant$ $\leqslant$ $\leqslant$

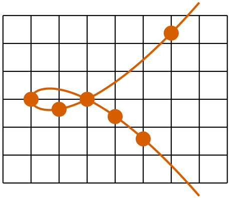


Intervals



Affine/linear sets

$\leqslant$



Algebraic sets = polynomial equalities

Linear system with rounding

Rounding: $\lfloor \cdot \rceil$ = round to nearest integer

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathbb{Q}^{2\times 2}, \qquad \left\lfloor \begin{pmatrix} x \\ y \end{pmatrix} \right\rceil = \begin{pmatrix} \lfloor x \rceil \\ \lfloor y \rceil \end{pmatrix}$$

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Problem: given $X_0 \in \mathbb{Q}^2$, define $X_{n+1} = \lfloor AX_n\rceil$

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- is $(X_n)_n$ eventually periodic?
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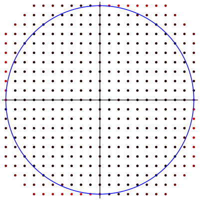

$r = 10, \theta = \pi/42$

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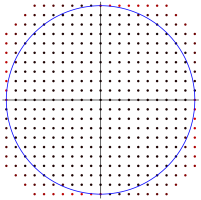

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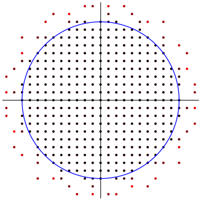

$r = 10, \theta = \frac{2^{0.4}\pi}{10}$

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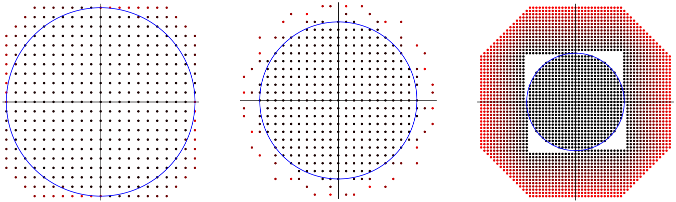

$r = 10, \theta = \pi/42$ $\quad r = 10, \theta = \frac{2^{0.4}\pi}{10}$ $\quad r = 15, \theta = \pi/91$

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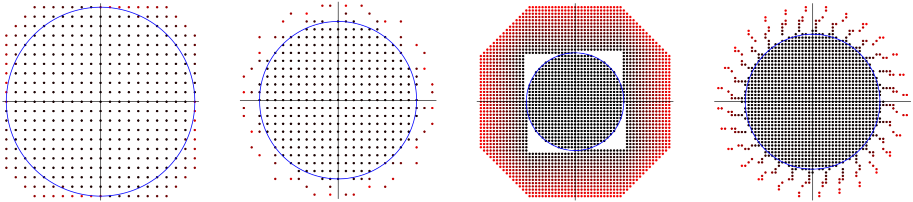

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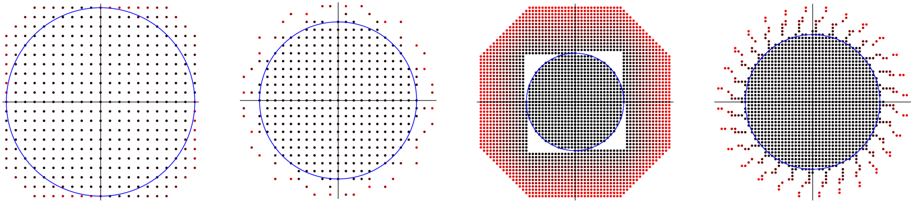

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Open problems! Only known for a few specific values of θ.

Linear dynamical systems are ubiquitous...

... and lead to very interesting mathematics!

Interesting related mathematics

- Linear recurrent sequences (LRS)

$$x_{n+k} = a_{k-1} a_{n+k-1} + \cdots + x_0 x_n$$

Fibonacci: $F_{n+2} = F_{n+1} + F_n$

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$$f(t) = P_1(t)e^{\lambda_1 t} + \cdots + P_n(t)e^{\lambda_n t}$$

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Reachability often harder/reduces to of these problems!

Algebraic numbers and conjectures

Algebraic number: root of polynomial with integer coefficients
Transcendental number: not algebraic, e.g. e, π

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If a, b are algebraic numbers with $a \neq 0, 1$ and b irrational, then (any value of) a^b transcendental.

Example: $2^{\sqrt{2}}$ is transcendental.

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Why is this related to reachability?

- target is usually rational/algebraic
- reachability creates constraints between numbers

Example: given $a, b \in \mathbb{Q}$, $P \in \mathbb{Q}[X]$ polynomial, find t such that

$$P(t) = a \quad \text{and} \quad e^t = b$$

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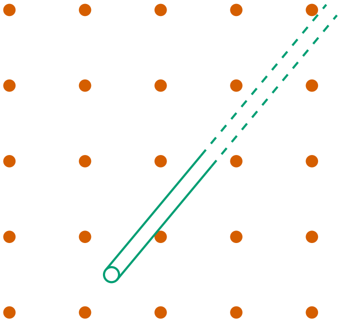
Biggest open question in this field: Schanuel's conjecture

Many problems boil down to diophantine equations/approximations:

- Finding integer points in cones: Kronecker's theorem



Transcendental number theory

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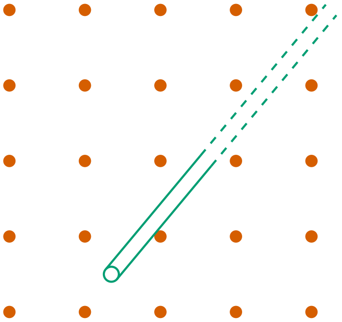

- Compare linear forms in logarithms: Baker's theorem

$$\sqrt{2} + \log\sqrt{3} - 3\log\sqrt{7} \quad \stackrel{?}{=} \quad 1 + \log 9 - \log \sqrt[42]{666}$$

(Semi-)group theory

Finitely generated matrix semigroup:
$A_1, \ldots, A_k \in \mathbb{Q}^{n\times n}$ generate a semigroup $S = \langle A_1, \ldots, A_k \rangle$

Example: $\mathsf{SL}_2(\mathbb{Z}) = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle$

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Undecidable in general, many decidable subclasses are known.
Equivalent to reachability of affine programs.

Algebraic geometry

Study systems of multivariate polynomial equations using abstract algebraic techniques, with applications to geometry.

Examples

$$
\begin{array}{rcl}
x^2 + y^2 + z^2 - 1 = 0 & \rightsquigarrow & \text{sphere in } \mathbb{R}^3 \\
x^2 + y^2 + z^2 = 1 \ \wedge \ x + y + z = 1 & \rightsquigarrow & \text{“sliced” sphere in } \mathbb{R}^3 \\
x^2 + 1 = 0 & \rightsquigarrow & \varnothing \text{ in } \mathbb{R} \\
x^2 + 1 = 0 & \rightsquigarrow & \{i, -i\} \text{ in } \mathbb{C}
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$$

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The field $\mathbb{K}$ is very important:

- real algebraic geometry: more “intuitive” but more difficult, really requires the study of *semi-algebraic sets*
- mainstream algebraic geometry: $\mathbb{K}$ is algebraically closed†, e.g. $\mathbb{C}$

† $\mathbb{K}$ is algebraically closed if every non-constant polynomial has a root in $\mathbb{K}$.

First-order theory of the reals

Many questions expressible in first-order logical theories:

- $\mathfrak{R}_0 = (\mathbb{R}, 0, 1, <, +, \cdot)$: decidable

$$\forall x, y \in \mathbb{R} \; \frac{x+y}{2} \geqslant \sqrt{xy}$$

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- $\mathfrak{R}_{\exp} = (\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos\restriction_{[0,1]})$: decidable subject to Schanuel's conjecture

$$\forall x \in \mathbb{R}\ x \neq 0 \Rightarrow t + te^{t} - 43e^{3t} \neq 1$$

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- Presburger arithmetic $(\mathbb{N}, 0, 1, <, +)$: decidable

$$\exists x \in \mathbb{N}^n\ Ax \geqslant b$$

Summary

Linear dynamical systems are ubiquitous and exact reachability questions lead to very interesting mathematical and logical questions.

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- exact reachability is not the only approach

  testing, probabilistic model checking, incomplete algorithms, ...