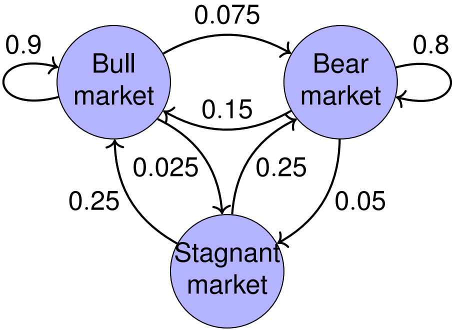
Linear Dynamical Systems

Reachability

Amaury Pouly

Examples: while loop, Markov chain



State: $X = (p_{bull}, p_{bear}, p_{stag}) \in [0,1]^3$

Transitions:

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$

$\rightarrow$ Linear dynamical system

$$X_{n+1} = AX_n$$

Linear loop

```
$p_{bull} := 0$
$p_{bear} := 1$
$p_{stag} := 0$
while $p_{bull} \leqslant 1/2$ do
    $\begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix} := A \begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix}$
```

The loop terminates if and only if the probability of a bull market is $> 1/2$.

Termination Linear Loops

Does this loop terminate?

Linear Loop

```
$x := 2^{-10}, y := 1$
until $\phi(x)$ do
    $\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
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$\rightsquigarrow$

Reachability problem

Given

- initial point: $x_0 \in \mathbb{Q}^d$,
- transition matrix: $A \in \mathbb{Q}^{d \times d}$,
- target set: $\mathcal{S} \subseteq \mathbb{R}^d$

decide if $\exists n \in \mathbb{N}.\ A^n x_0 \in \mathcal{S}$.

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$x := 2^{-10}, y := 1$

until $x = 42$ and $y = 36$ do

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Natural choices for $\mathcal{S}$:

- point:

$$\exists n \in \mathbb{N}\ A^n x_0 = y$$

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- affine subspace:

$$\exists n \in \mathbb{N}\ MA^n x_0 = b$$

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until $x \geqslant y$ do

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- (semi-)algebraic sets

$$\exists n \in \mathbb{N}\ p(A^n x_0) \geqslant 0$$

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until $x^2 y \geqslant 1$ or $x = y$ do

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- (semi-)algebraic sets

$$\exists n \in \mathbb{N}\ p(A^n x_0) \geqslant 0$$

- boolean combinations
- replace x_0 by an initial set $\mathcal{X}$

$$\exists x_0 \in \mathcal{X} \exists n \in \mathbb{N}\ A^n x_0 \in \mathcal{S}$$

$$\forall x_0 \in \mathcal{X} \exists n \in \mathbb{N}\ A^n x_0 \in \mathcal{S}$$

What is decidable about linear loops?

Problem: given x_0, A and $\mathcal{S}$, decide if $\exists n \in \mathbb{N}$ such that $A^n x_0 \in \mathcal{S}$.

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Decidable in polynomial time when $\mathcal{S}$ is a singleton.

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Problem: given $\mathcal{X}$, A and $\mathcal{S}$, decide if $\exists n \in \mathbb{N}$ such that $A^n \mathcal{X} \cap \mathcal{S} \neq \varnothing$.

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Why do we need the dimension to be small?

From loops to recurrent sequences

Linear Loop

```
$x := x_0$
until $3x_1 - 7x_2 + 4x_3 = 0$ do
$x := Ax$
```

From loops to recurrent sequences

Linear Loop

$x := x_0$
until $y^T x = 0$ do $x := Ax$

$\rightsquigarrow$

Half-space reachability

Given $x, y \in \mathbb{Q}^d$, $A \in \mathbb{Q}^{d \times d}$,
decide if $\exists n \in \mathbb{N}.\, y^T A^n x_0 = 0$.

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Given $x, y \in \mathbb{Q}^d$, $A \in \mathbb{Q}^{d \times d}$,
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Consider the sequence $u_n = y^T A^n x$.

Lemma

There exists $a_0, \ldots, a_{d-1} \in \mathbb{Q}$ *such that*

$$u_{n+d} = a_{d-1} u_{n+d-1} + \cdots + a_0 u_n, \qquad \forall n \in \mathbb{N}.$$

In other words, $(u_n)_n$ is a linear recurrent sequence (LRS).

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- Fibonacci: $F_{n+2} = F_{n+1} + F_n$
- Pell numbers: $P_{n+2} = 2P_{n+1} + P_n$
- very common in combinatorics

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Lemma

For any LRS $(u_n)_n$, there exists x_0, y and A such that $u_n = y^T A^n x_0$.

Skolem and posivity problems

Linear recurrent sequence (LRS) of order d:

$$u_{n+d} = a_{d-1}u_{n+d-1} + \cdots + a_0 u_n, \qquad \forall n \in \mathbb{N}.$$

Remark: entirely determined by $u_0, \ldots, u_{d-1}$ and $a_0, \ldots, a_{d-1}$

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Given a LRS $(u_n)_n$, decide if $u_n = 0$ for some $n \in \mathbb{N}$.

This problem has been open for 70 years!

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Positivity Problem

Given a LRS $(u_n)_n$, decide if $u_n \geqslant 0$ for all $n \in \mathbb{N}$.

Harder than Skolem

Skolem-Mahler-Lech theorem

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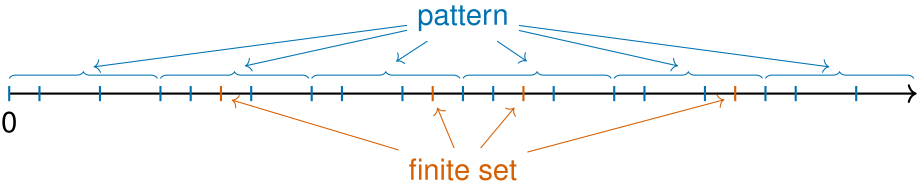
Given a LRS $(u_n)_n$, decide if $u_n = 0$ for some $n \in \mathbb{N}$.

Theorem (Skolem, Mahler, and Lech, 1933, 1953, 1957)

The set $\{n \in \mathbb{N} : u_n = 0\}$ *is a union of finitely arithmetic progression and a finite set.*



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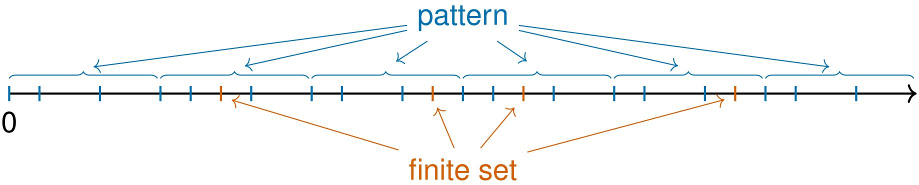

The regular patterm is computable. Nothing is known about the finite set: the proof is nonconstructive and uses p-adic analysis.

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Theorem (Mignotte, Shorey, Tijdeman; Vereshchagin, 1985)

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How can we show hardness without proving undecidability?

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For any $x \in \mathbb{R}$, the (homogeneous Diophantine approximation) type

$$L(x) = \inf \left\{ c \in \mathbb{R} : \left| x - \frac{n}{m} \right| < \frac{c}{m^2} \text{ for some } n, m \in \mathbb{Z} \right\}.$$

Intuitively, if $L(x) > 0$ then x is badly approximable by rationals.

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Intuitively, if $L(x) > 0$ then x is badly approximable by rationals. Almost nothing known for any concrete x except that $L(x) \in [0, 1/\sqrt{5}]$.

Theorem (Ouaknine and Worrell, 2013)

If Skolem is decidable at order 5 then one can approximate $L(x)$ *with arbitrary precision for a large class of numbers* x*.*

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Theorem (Ouaknine and Worrell, 2014)

The ultimate positivity problem is decidable for simple[*] LRS. It is at least as hard as deciding $\exists\mathbb{R}$*.*

---

[*] The associated characteristic polynomial has no repeated roots.

First-order queries on orbits

First-order orbit query (FOOQ): fully quantified first-order sentence whose atomic proposition are of the form

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Theorem (Almagor, Ouaknine and Worrell, 2021)

Given A and $\Phi(n)$ a FOOQ, it is decidable whether $\exists n \in \mathbb{N}.\, \Phi(n)$ in dimension $\leqslant 3$.

MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets.

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Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^{\mathbb{N}}$ by

$$w_n = \big(A^n x \in \mathcal{T}_1, \ldots, A^n x \in \mathcal{T}_k\big).$$

Intuition: w_n records to which sets $A^n x$ belongs to at eact step n.

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Problem: given an MSO formula Ψ over $(\mathbb{N}, <)$, decide whether $w \models \Psi$.

Examples: $P_i(n)$ means $A^n x \in \mathcal{T}_i$

- $\mathcal{T}_i$ is reachable: $\exists n.\, P_i(n)$
- whenever $\mathcal{T}_i$ is visited $\mathcal{T}_j$ is visited some point later:

$$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$

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$$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$

- in target $\mathcal{T}_i$ at every odd position:

$$\exists O \subseteq \mathbb{N} : \boxed{\text{formula to define odd numbers}} \wedge \forall x : x \in O \Rightarrow P_i(x)$$

MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets.
Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^{\mathbb{N}}$ by

$$w_n = \big(A^n x \in \mathcal{T}_1, \ldots, A^n x \in \mathcal{T}_k\big).$$

Intuition: w_n records to which sets $A^n x$ belongs to at eact step n.
Problem: given an MSO formula Ψ over $(\mathbb{N}, <)$, decide whether $w \models \Psi$.

Theorem (Karimov, Lefaucheux, Ouaknine, Purser, Varonka, Whiteland, Worrell)

This is decidable if all $\mathcal{T}_i$ either have intrinsic dimension 1 or are included in a subspace of dimension 3.

Examples: $P_i(n)$ means $A^n x \in \mathcal{T}_i$

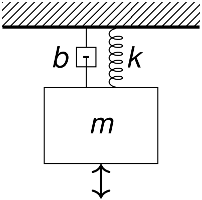
- $\mathcal{T}_i$ is reachable: $\exists n.\, P_i(n)$
- whenever $\mathcal{T}_i$ is visited $\mathcal{T}_j$ is visited some point later:

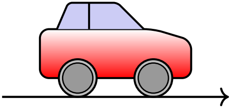
$$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$

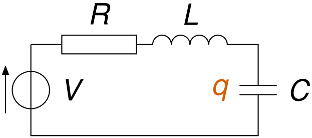
- in target $\mathcal{T}_i$ at every odd position:

$$\exists O \subseteq \mathbb{N} : \boxed{\text{formula to define odd numbers}} \wedge \forall x : x \in O \Rightarrow P_i(x)$$

Continuous linear dynamical systems







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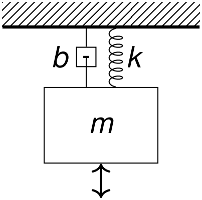
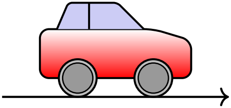
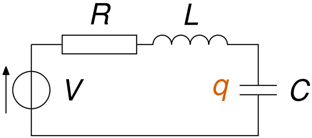






Linear differential equation:

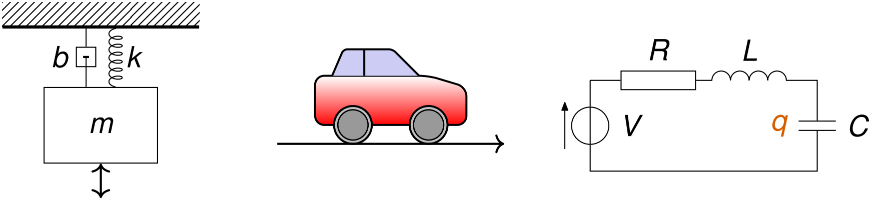
$$x'(t) = Ax(t) \qquad x(0) = x_0$$

Example:

$$x'(t) = 7x(t)$$

$$\rightsquigarrow x(t) = e^{7t}$$

Continuous linear dynamical systems



Linear differential equation:

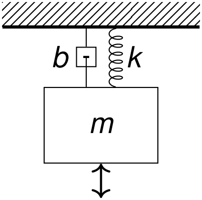
$$x'(t) = Ax(t) \qquad x(0) = x_0$$

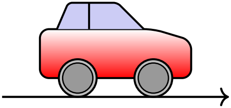
Example:

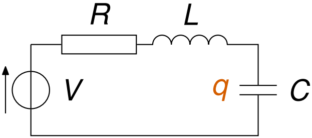
$$x'(t) = 7x(t) \qquad \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -x_1(t) \end{cases} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rightsquigarrow x(t) = e^{7t} \qquad \rightsquigarrow \begin{cases} x_1(t) = \sin(t) \\ x_2(t) = \cos(t) \end{cases}$$

Continuous linear dynamical systems







Linear differential equation:

$$x'(t) = Ax(t) \qquad x(0) = x_0$$

General solution form:

$$x(t) = e^{At} x_0$$

$$\text{where } e^{M} = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

Continuous reachability

Continuous Skolem problem

Given x, y and A, decide if $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

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Continuous positivity Problem

Given x, y and A, decide whether $x^T e^{At} y \geqslant 0$ for all $t \geqslant 0$.

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Continuous positivity Problem

Given x, y and A, decide whether $x^T e^{At} y \geqslant 0$ for all $t \geqslant 0$.

Continuous positivity is inter-reducible with continuous Skolem.

The decidability of all these problems is also open!

A link with number theory

Some reachability questions look like this :

$$\exists t \in \mathbb{R}.\ 42t^7 = 56 \ \wedge \ e^{3t} - e^t = 9$$

A link with number theory

Some reachability questions look like this (P, Q polynomials):

$$\exists t \in \mathbb{R}.\ P(t) = 0 \wedge Q(e^t) = 0$$

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If t is a nonzero algebraic number then e^t is transcendental.

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Theorem (Special case of Lindemann–Weierstrass)

If t is a nonzero algebraic number then e^t is transcendental.

- $P(t) = 0$ so t is algebraic (by definition)
- Lindemann–Weierstrass: e^t transcendental (unless $t = 0$)
- hence $Q(e^t) \neq 0$ (except maybe if $t = 0$)

Exponential polynomial

In general,

$$x^T e^{At} y = \sum_{i=1}^{d} P_i(t) e^{\lambda_i t}$$

where P_i polynomial, $\lambda_i \in \mathbb{C}$ eigenvalues of A.

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Theorem (Wilkie and MacIntyre)

If Schanuel's conjecture is true, then, for each $k \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos\restriction_{[0,k]}, \sin\restriction_{[0,k]})$ is decidable.

- algorithm always correct, only termination requires the conjecture

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What is Schanuel's conjecture?

Schanuel's conjecture

Schanuel's conjecture

If $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$, then at least n numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are algebraically independent.

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Example: π and e are algebraically independent

$$z_1 = i\pi, z_2 = 1 \qquad \rightsquigarrow \qquad e^{z_1} = -1, e^{z_2} = e.$$

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Clearly z_1 and z_2 are linearly independent over $\mathbb{Q}$. So at least 2 of $i\pi, 1, -1, e$ are algebraically independent. But 1 is algebraic so π and e are algebraically independent.

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Summary:

- Schanuel implies that π, e, $\pi + e$, $e\pi$, ... are transcendental.
- π and e are known to be transcendental
- $\pi + e$ is **not known** to be transcendental

Continuous reachability

Bounded continuous Skolem problem: given x, y and A, decide if

- unbounded: $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$.
- bounded: $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

Theorem (Chonev, Ouaknine and Worrell, 2016)

The bounded continuous Skolem Problem is decidable subject to Schanuel's conjecture.

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Theorem (Chonev, Ouaknine and Worrell, 2016)

The bounded continuous Skolem Problem is decidable subject to Schanuel's conjecture.

Theorem (Chonev, Ouaknine and Worrell, 2016)

If the (unbounded) continuous Skolem Problem is decidable then the Diophantine-approximation types of all real algebraic numbers is computable.

In other words: it requires new mathematics...

More complicated programs

Linear loop with if

```
$x := 2^{-10}$
$y := 1$
while $y \geqslant x$ do
    if $y \geqslant 2x$ then
        $\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
    else
        $\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
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$\rightsquigarrow$

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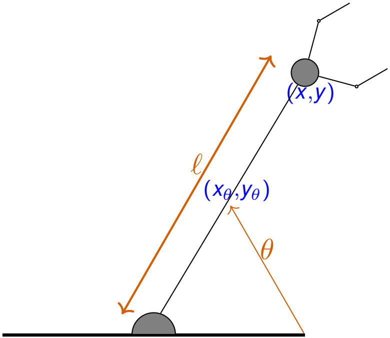
Reachability is trivially undecidable by simulating two counter automata

- Overapproximate behaviours
- Nondeterminic

Example: 2D robot



State: $\vec{u} = (x_\theta, y_\theta, x, y)$

Discretized actions:

- rotate arm by ψ
- change arm length by δ

$\leadsto$ Linear transformations

Rotate arm by ψ:

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} \leftarrow \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix}$$

Change arm length by δ:

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} x \\ y \end{pmatrix} + \delta \begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix}$$

Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$?

Example: $\exists n \in \mathbb{N}$ such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \quad ?$$

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Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices

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Input: $A, B, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$?

Example: $\exists n, m \in \mathbb{N}$ such that

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \quad ?$$

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Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^{k} A_i^{n_i} = C \quad ?$

Example: $\exists n, m, p \in \mathbb{N}$ such that

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^m \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}^p = \begin{bmatrix} 81 & 260 \\ 0 & 1 \end{bmatrix} ?$$

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Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d\times d}$ matrices
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Semigroup: $\langle A_1, \ldots, A_k\rangle$ = all finite products of $A_1, \ldots, A_k$
Examples:

$$A_1 A_3 A_2 \quad A_1 A_2 A_1 A_2 \quad A_3^8 A_2 A_1^3 A_3^{42}$$

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Discrete reachability problems

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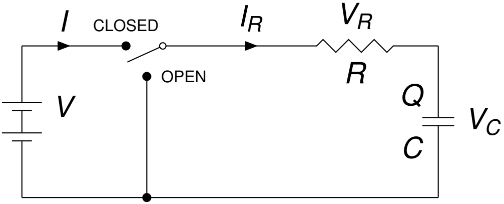
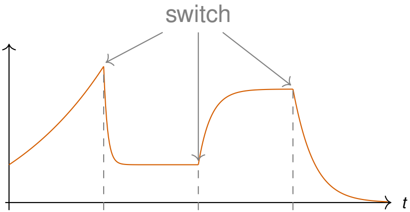
Discrete reachability problems

Every nontrivial extension of simple linear loops seems to lead to undecidable problems. What about the continuous setting?

RC circuit





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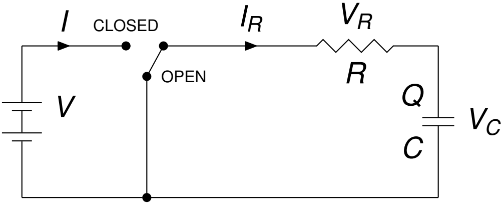
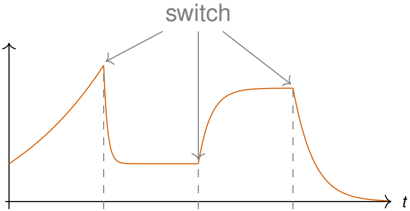
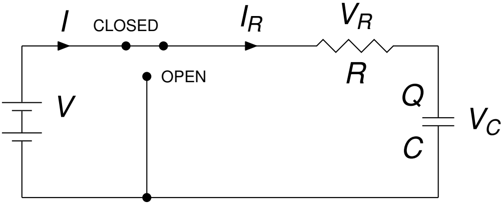


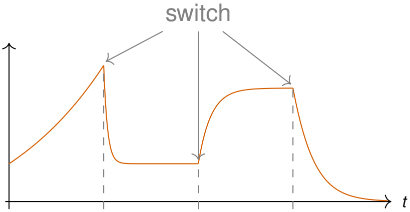


OPEN

$$\boxed{\begin{aligned} \dot{I} &= 0 \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}}$$

RC circuit





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CLOSED

$$\boxed{\begin{aligned} \dot{I} &= -\frac{1}{RC} I_R \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}}$$

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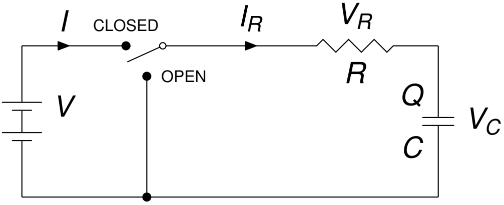

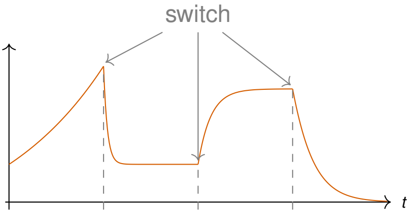

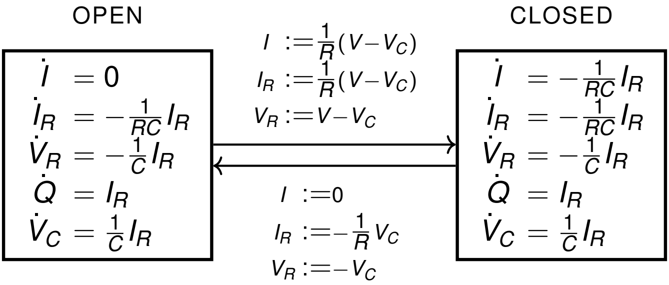

OPEN

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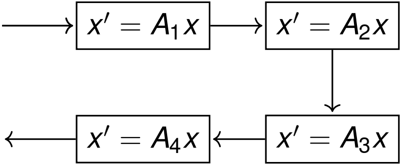
$$\begin{aligned} I &:= \frac{1}{R}(V - V_C) \\ I_R &:= \frac{1}{R}(V - V_C) \\ V_R &:= V - V_C \end{aligned}$$

$$\begin{aligned} I &:= 0 \\ I_R &:= -\frac{1}{R} V_C \\ V_R &:= -V_C \end{aligned}$$

CLOSED

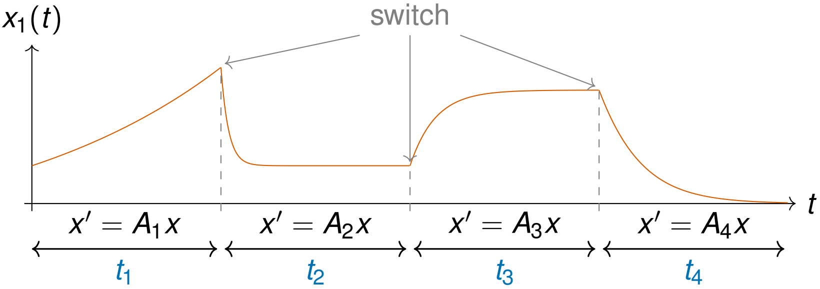
$$\begin{aligned} \dot{I} &= -\frac{1}{RC} I_R \\ \dot{I}_R &= -\frac{1}{RC} I_R \\ \dot{V}_R &= -\frac{1}{C} I_R \\ \dot{Q} &= I_R \\ \dot{V}_C &= \frac{1}{C} I_R \end{aligned}$$

Switching systems

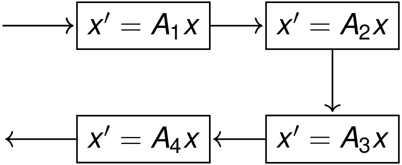


Restricted hybrid system:

- linear dynamics
- no guards (nondeterministic)
- no discrete updates



Switching systems



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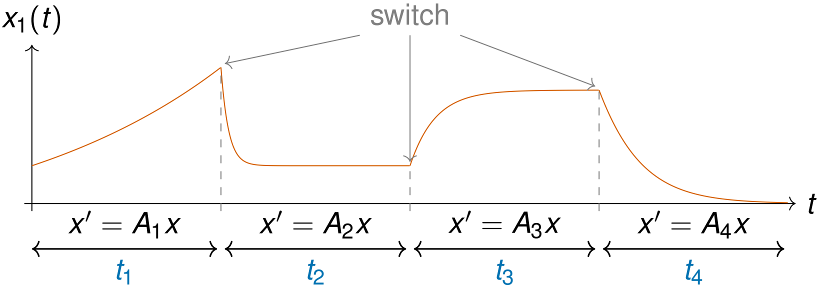
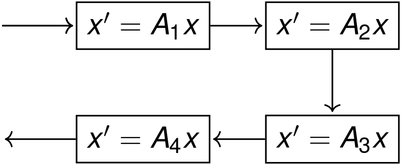


Dynamics:

$$e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1}$$

Switching systems



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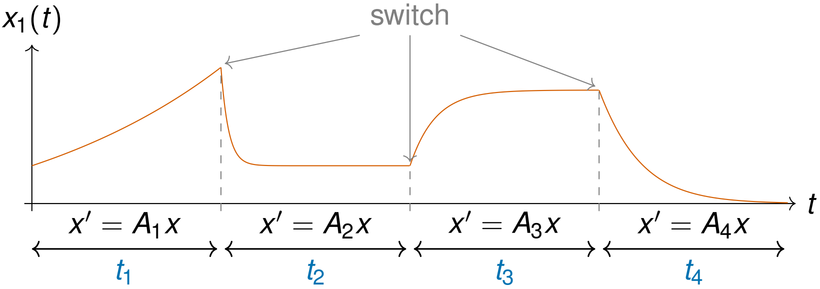

Problem:

$$e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} = C \quad ?$$

What we control: $t_1, t_2, t_3, t_4 \in \mathbb{R}_{\geqslant 0}$

Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d\times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C \quad ?$

Example: $\exists t \in \mathbb{R}$ such that

$$\exp\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t\right) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \quad ?$$

Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$?

✓ Decidable (PTIME)

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Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
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Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At} e^{Bu} = C \quad ?$

Example: $\exists t, u \in \mathbb{R}$ such that

$$\exp\left(\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} t\right) \exp\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u\right) = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \quad ?$$

Related work in the continuous case

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Input: $A, B, C \in \mathbb{Q}^{d\times d}$ matrices
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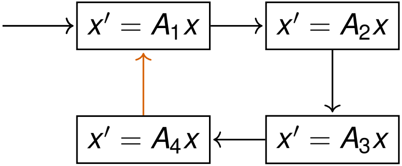
Switching system



$x' = A_1 x$

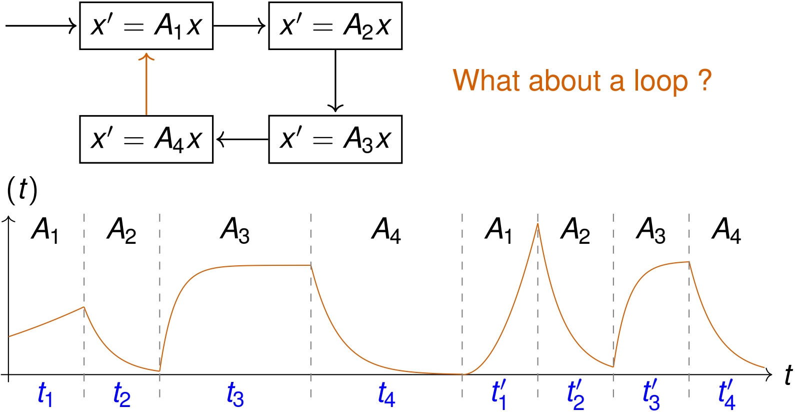
$x' = A_2 x$

$x' = A_3 x$

$x' = A_4 x$

What about a loop ?

Switching system

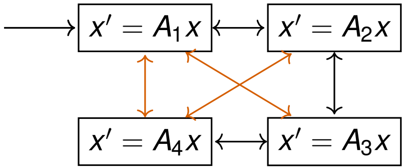


What about a loop ?

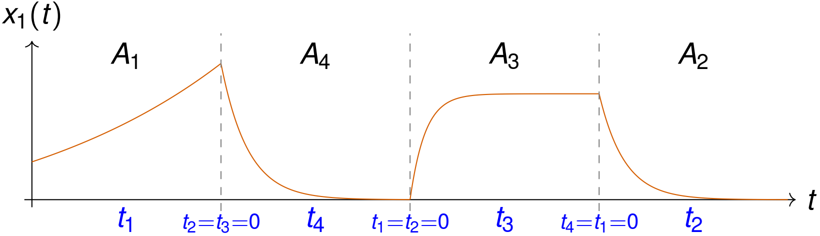
Dynamics:

$$e^{A_4 t'_4} e^{A_3 t'_3} e^{A_2 t'_2} e^{A_1 t'_1} e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1}$$

Switching system



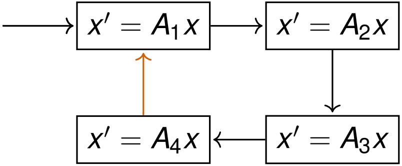
Loop ⇔ clique



Remark:

zero time dynamics ($t_i = 0$) are allowed

Switching system



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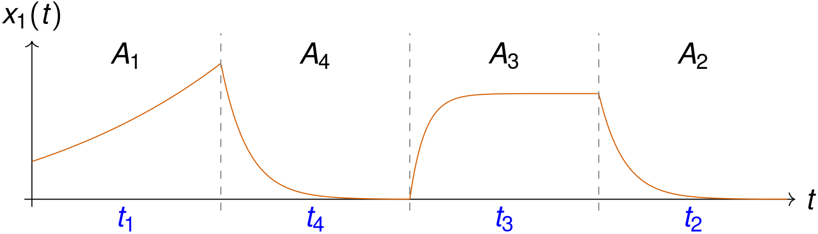
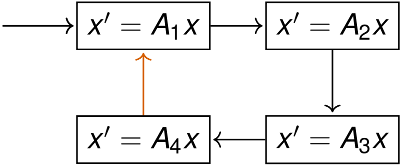


Dynamics:

any finite product of $e^{A_i t}$ $\rightsquigarrow$ semigroup!

Switching system



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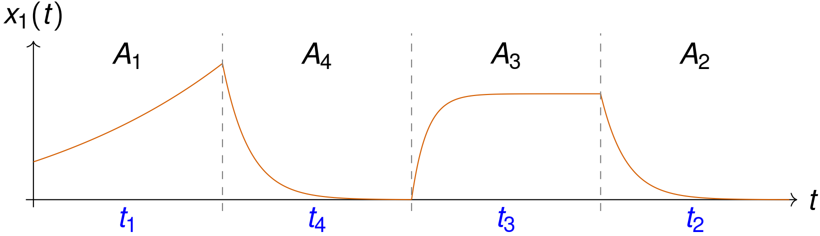


Problem:

$$C \in \mathcal{G} \quad ?$$

where $\mathcal{G} = \langle \text{semigroup generated by } e^{A_i t} \text{ for all } t \geqslant 0 \rangle$

Reachability for switching systems

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t_1, \ldots, t_k \geqslant 0$ such that

$$\prod_{i=1}^{n} e^{A_i t_i} = C \quad ?$$

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output:

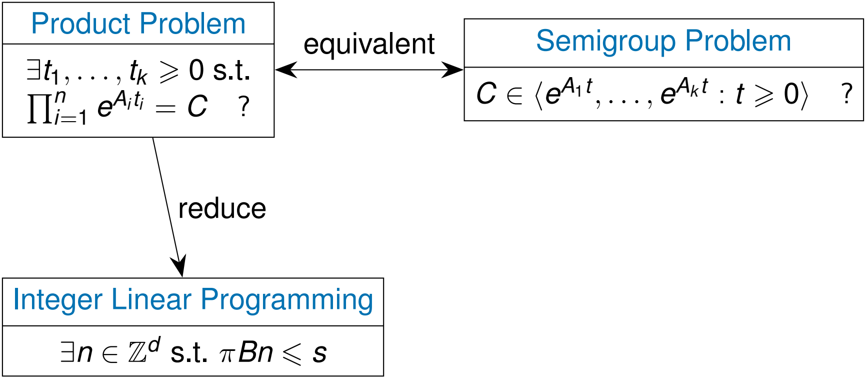
$$C \in \langle \text{semigroup generated by } e^{A_1 t}, \ldots, e^{A_k t} : t \geqslant 0 \rangle \quad ?$$

Theorem (Ouaknine, P, Sous-Pinto, Worrell)

Both problems are:

- *Undecidable in general*
- Decidable *when all the A_i commute*

Some words about the proof (commuting case)



Product Problem

$\exists t_1, \ldots, t_k \geqslant 0$ s.t. $\prod_{i=1}^{n} e^{A_i t_i} = C$?

equivalent

Semigroup Problem

$C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geqslant 0 \rangle$?

reduce

Integer Linear Programming

$\exists n \in \mathbb{Z}^d$ s.t. $\pi B n \leqslant s$

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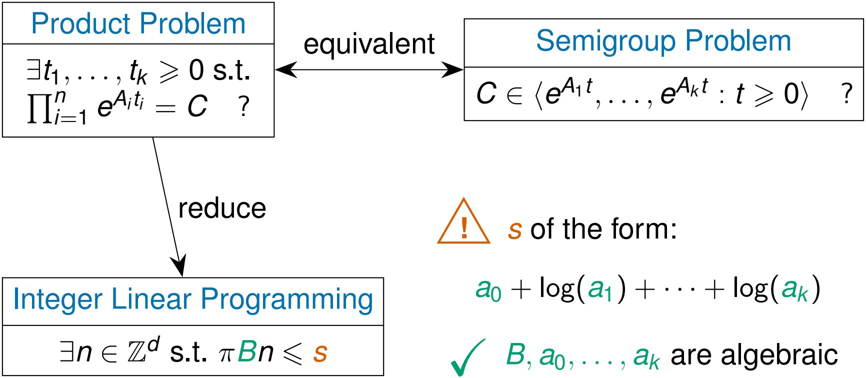

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⚠ s of the form:

$$a_0 + \log(a_1) + \cdots + \log(a_k)$$

✓ $B, a_0, \ldots, a_k$ are algebraic

Some words about the proof (commuting case)

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✓ $B, a_0, \dots, a_k$ are algebraic

How did we get from reals to integers with π ?

$$e^{it} = \alpha \quad \Leftrightarrow \quad t \in \log(\alpha) + 2\pi\mathbb{Z}$$

Integer Linear Programming

$$\exists n \in \mathbb{Z}^d \text{ such that } \pi B n \leqslant s \quad ?$$

where s is a linear form in logarithms of algebraic numbers

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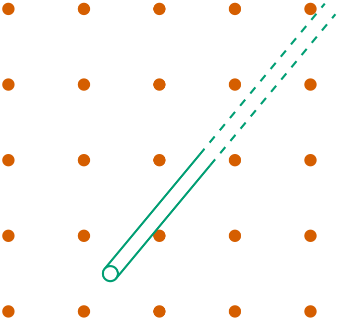
Key ingredient: Diophantine approximations

- Finding integer points in cones: Kronecker's theorem



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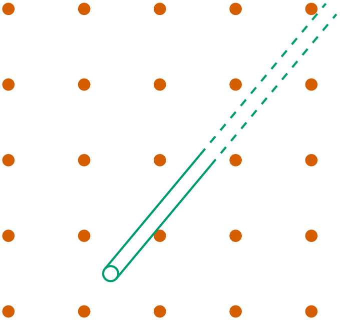
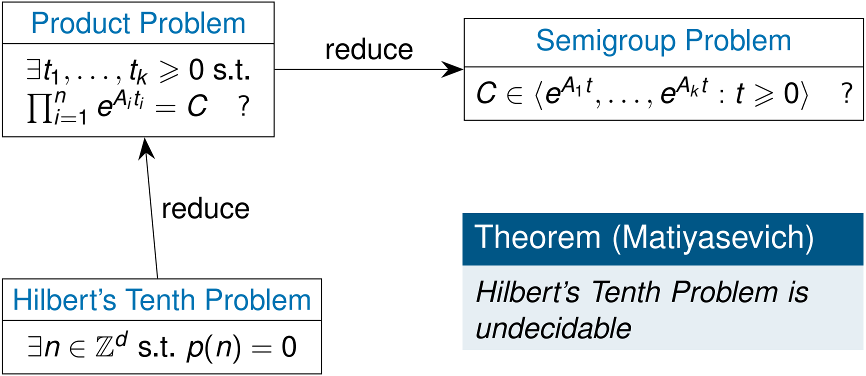


- Compare linear forms in logs: Baker's theorem

$$\sqrt{2} + \log\sqrt{3} - 3\log\sqrt{7} \quad \overset{?}{=} \quad 1 + \log 9 - \log \sqrt[42]{666}$$

Some words about the proof (general case)



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$\exists t_1, \ldots, t_k \geqslant 0$ s.t.
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reduce

Semigroup Problem

$C \in \langle e^{A_1 t}, \ldots, e^{A_k t} : t \geqslant 0 \rangle \quad ?$

reduce

Hilbert's Tenth Problem

$\exists n \in \mathbb{Z}^d$ s.t. $p(n) = 0$

Theorem (Matiyasevich)

Hilbert's Tenth Problem is undecidable

Summary on reachability

Exact reachability is hard:

- Skolem/Positivity problem for linear loops (Open for 70 years)
- Every mild extension is undecidable
- Decidability requires very strong assumptions (commuting matrices)

Continuous vs discrete setting

- similar results
- different techniques
- continuous setting can leverage powerful results/conjectures