Linear Dynamical Systems
Reachability

Amaury Pouly
Examples: while loop, Markov chain

**State:** $X = (p_{bull}, p_{bear}, p_{stag}) \in [0, 1]^3$

**Transitions:**

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$

→ Linear dynamical system

$$X_{n+1} = AX_n$$

**Linear loop**

$p_{bull} := 0$

$p_{bear} := 1$

$p_{stag} := 0$

while $p_{bull} \leq 1/2$ do

$$\begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix} := A \begin{bmatrix} p_{bull} \\ p_{bear} \\ p_{stag} \end{bmatrix}$$

The loop terminates if and only if the probability of a bull market is $> 1/2$. 
Termination Linear Loops

Does this loop terminate?

Linear Loop

\[
x := 2^{-10}, \quad y := 1
\]

until \( \phi(x) \) do

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} := \begin{bmatrix}
  2 & 0 \\
  \frac{7}{4} & \frac{1}{4}
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]
Termination Linear Loops

Does this loop terminate?

Linear Loop

\[
\begin{align*}
x &:= 2^{-10}, \quad y := 1 \\
\text{until } &\phi(x) \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} &:= \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Reachability problem

Given

- **initial point:** \( x_0 \in \mathbb{Q}^d \),
- **transition matrix:** \( A \in \mathbb{Q}^{d \times d} \),
- **target set:** \( S \subseteq \mathbb{R}^d \)

decide if \( \exists n \in \mathbb{N}. A^n x_0 \in S \).
Termination Linear Loops

Does this loop terminate?

**Linear Loop**

\[
\begin{align*}
x & := 2^{-10}, \quad y := 1 \\
\text{until } x &= 42 \text{ and } y = 36 \text{ do} \\
x_y & := \begin{bmatrix} 2 & 0 \\ 7/4 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

Natural choices for \( S \):

- **point:**
  \[
  \exists n \in \mathbb{N} \quad A^n x_0 = y
  \]

**Reachability problem**

**Given**

- initial point: \( x_0 \in \mathbb{Q}^d \),
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| $x := 2^{-10}, y := 1$
| until $x = y$ do
| $\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 7/4 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ |

Reachability problem

Given:
- initial point: $x_0 \in \mathbb{Q}^d$
- transition matrix: $A \in \mathbb{Q}^{d \times d}$
- target set: $S \subseteq \mathbb{R}^d$

decide if $\exists n \in \mathbb{N}. A^n x_0 \in S$.

Natural choices for $S$:
- point:
  $\exists n \in \mathbb{N}. A^n x_0 = y$
- affine subspace:
  $\exists n \in \mathbb{N}. MA^n x_0 = b$
Termination Linear Loops

Does this loop terminate?

Linear Loop

\[ x := 2^{-10}, y := 1 \]

until \( x \geq y \) do

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} := \begin{bmatrix}
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Does this loop terminate?

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\[
\begin{align*}
x & := 2^{-10}, 
\ y & := 1 \\
\text{until } x^2 y & \geq 1 \text{ do} \\
\begin{bmatrix} x \\ y \end{bmatrix} & := \begin{bmatrix} 2 & 0 \\ 7/4 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
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decide if \( \exists n \in \mathbb{N}. A^n x_0 \in S \).

- (semi-)algebraic sets
  \[ \exists n \in \mathbb{N} \ p(A^n x_0) \geq 0 \]
Termination Linear Loops

Does this loop terminate?

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\[
x := 2^{-10}, \quad y := 1
\]

until \(x^2y \geq 1\) or \(x = y\) do

\[
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
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  \(\exists n \in \mathbb{N} p(A^n x_0) \geq 0\)

- boolean combinations
Termination Linear Loops

Does this loop terminate?

**Linear Loop**

\[
\begin{align*}
x & \in [0, 1], \\
y & \in [1, 2]
\end{align*}
\]

until \( \phi(x) \) do

\[
\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
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- (semi-)algebraic sets
  \( \exists n \in \mathbb{N}. p(A^n x_0) \geq 0 \)
- boolean combinations
- replace \( x_0 \) by an initial set \( \mathcal{X} \)
  \( \exists x_0 \in \mathcal{X}. \exists n \in \mathbb{N}. A^n x_0 \in S \)
  \( \forall x_0 \in \mathcal{X}. \exists n \in \mathbb{N}. A^n x_0 \in S \)
What is decidable about linear loops?

**Problem:** given $x_0$, $A$ and $S$, decide if $\exists n \in \mathbb{N}$ such that $A^n x_0 \in S$. 

Theorem (Orbit problem; Kannan and Lipton 1980, 1986)

Decidable in polynomial time when $S$ is a singleton.

Already nontrivial proof using algebraic number theory!

Theorem (Chonev, Ouaknine and Worrell, 2016)

Decidable (in $\text{NP} \cap \text{RP}$) when $S$ is a linear subspace of dimension $\leq 3$.

Decidable (in $\text{PSPACE}$) when $S$ is a polytope of dimension $\leq 3$.

**Problem:** given $X$, $A$ and $S$, decide if $\exists n \in \mathbb{N}$ such that $A^n X \cap S \neq \emptyset$.

Theorem (Almagor, Ouaknine and Worrell, 2017)

Decidable (in $\text{PSPACE}$) when $X$, $S$ are polytopes of dimension $\leq 3$.

Why do we need the dimension to be small?
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Why do we need the dimension to be small?
From loops to recurrent sequences

**Linear Loop**

\[
x := x_0 \\
\text{until } 3x_1 - 7x_2 + 4x_3 = 0 \text{ do} \\
x := Ax
\]
From loops to recurrent sequences

**Linear Loop**

\[ x := x_0 \]

until \( y^T x = 0 \) do \( x := Ax \)

**Half-space reachability**

Given \( x, y \in \mathbb{Q}^d, A \in \mathbb{Q}^{d \times d} \),

decide if \( \exists n \in \mathbb{N}. y^T A^n x_0 = 0 \).
From loops to recurrent sequences

**Linear Loop**

\[
x := x_0 \quad \text{until } y^T x = 0 \quad \text{do } x := Ax
\]

**Half-space reachability**

Given \( x, y \in \mathbb{Q}^d, A \in \mathbb{Q}^{d \times d} \), decide if \( \exists n \in \mathbb{N}. y^T A^n x_0 = 0 \).

Consider the sequence \( u_n = y^T A^n x \).

**Lemma**

There exist \( a_0, \ldots, a_{d-1} \in \mathbb{Q} \) such that

\[
u_{n+d} = a_{d-1} u_{n+d-1} + \cdots + a_0 u_n, \quad \forall n \in \mathbb{N}.
\]

In other words, \( (u_n)_n \) is a linear recurrent sequence (LRS).
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- **Fibonacci**: \( F_{n+2} = F_{n+1} + F_n \)
- **Pell numbers**: \( P_{n+2} = 2P_{n+1} + P_n \)
- very common in combinatorics
From loops to recurrent sequences

Linear Loop

\[
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\]

Half-space reachability

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decide if \( \exists n \in \mathbb{N}. y^T A^n x_0 = 0 \).

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Lemma

There exists \( a_0, \ldots, a_{d-1} \in \mathbb{Q} \) such that

\[
\begin{align*}
  u_{n+d} &= a_{d-1} u_{n+d-1} + \cdots + a_0 u_n, & \forall n \in \mathbb{N}.
\end{align*}
\]

In other words, \((u_n)_n\) is a linear recurrent sequence (LRS). Conversely,

Lemma

For any LRS \((u_n)_n\), there exists \( x_0, y \) and \( A \) such that \( u_n = y^T A^n x_0 \).
Linear recurrent sequence (LRS) of order $d$:

$$u_{n+d} = a_{d-1}u_{n+d-1} + \cdots + a_0 u_n, \quad \forall n \in \mathbb{N}.$$  

**Remark:** entirely determined by $u_0, \ldots, u_{d-1}$ and $a_0, \ldots, a_{d-1}$
Skolem and positivity problems

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Skolem Problem

Given a LRS $(u_n)_n$, decide if $u_n = 0$ for some $n \in \mathbb{N}$.

This problem has been open for 70 years!
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<th><strong>Positivity Problem</strong></th>
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Harder than Skolem
Skolem-Mahler-Lech theorem

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Theorem (Skolem, Mahler, and Lech, 1933, 1953, 1957)

The set \(\{ n \in \mathbb{N} : u_n = 0 \}\) is a union of finitely arithmetic progression and a finite set.
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Theorem (Skolem, Mahler, and Lech, 1933, 1953, 1957)

The set \(\{ n \in \mathbb{N} : u_n = 0 \}\) is a union of finitely arithmetic progression and a finite set.

The regular pattern is computable. Nothing is known about the finite set: the proof is nonconstructive and uses \(p\)-adic analysis.
Skolem in low dimension

Theorem (Mignotte, Shorey, Tijdeman; Vereshchagin, 1985)

The Skolem problem is decidable for LRS of order 4.

Theorem (Blondel and Portier, 2002)

The Skolem problem is NP-hard.

For any \( x \in \mathbb{R} \), the (homogeneous Diophantine approximation) type

\[
L(x) = \inf_{n, m} c \in \mathbb{R} : \left| x - \frac{n}{m} \right| < c m^2
\]

for some \( n, m \in \mathbb{Z} \).

Intuitively, if \( L(x) > 0 \) then \( x \) is badly approximable by rationals.

Almost nothing known for any concrete \( x \) except that \( L(x) \in [0, 1/\sqrt{5}] \).

Theorem (Ouaknine and Worrell, 2013)

If Skolem is decidable at order 5 then one can approximate \( L(x) \) with arbitrary precision for a large class of numbers \( x \).
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For any $x \in \mathbb{R}$, the (homogeneous Diophantine approximation) type $L(x) = \inf_{n, c \in \mathbb{R}}: x - n^m < c^m$ for some $n, m \in \mathbb{Z}$.

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How can we show hardness without proving undecidability?
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If Skolem is decidable at order 5 then one can approximate $L(x)$ with arbitrary precision for a large class of numbers $x$. 
Positivity and eventual positivity

Positivity Problem

Given a LRS \((u_n)_n\), decide if \(u_n \geq 0\) for all \(n \in \mathbb{N}\).

Theorem (Laohakosol and Tangsupphathawat, 2009)

The positivity problem is decidable at order 3.

Ultimate positivity Problem

Given a LRS \((u_n)_n\), decide if \(\exists N \in \mathbb{N}, \text{ such that } u_n \geq 0 \text{ for all } n \geq N\).

Theorem (Ouaknine and Worrell, 2014)

The ultimate positivity problem is decidable for simple \(*\) LRS. It is at least as hard as deciding \(\exists R\).
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*The ultimate positivity problem is decidable for simple* LRS. It is at least as hard as deciding \(\exists R\).

*The associated characteristic polynomial has no repeated roots.*
First-order queries on orbits

First-order orbit query (FOOQ): fully quantified first-order sentence whose atomic proposition are of the form

\[ p(x) \geq 0, \quad A^n x \in T \quad (T \text{ semialgebraic set}). \]
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- \( A^n S \cap T \neq \emptyset: \exists x \in \mathbb{R}^d. x \in S \land A^n x \in T \)
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- $A^n S \cap T \neq \emptyset : \exists x \in \mathbb{R}^d. x \in S \land A^n x \in T$
- $A^n S \subseteq T : \forall x \in \mathbb{R}^d. x \in S \rightarrow A^n x \in T$
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Examples: \( \exists n \in \mathbb{N} \) such that...

- \( A^n x = y : A^n x \in \{y\} \)
- \( A^n S \cap T \neq \emptyset : \exists x \in \mathbb{R}^d. x \in S \land A^n x \in T \)
- \( A^n S \subseteq T : \forall x \in \mathbb{R}^d. x \in S \rightarrow A^n x \in T \)

Theorem (Almagor, Ouaknine and Worrell, 2021)

*Given A and \( \Phi(n) \) a FOOQ, it is decidable whether \( \exists n \in \mathbb{N}. \Phi(n) \) in dimension \( \leq 3 \).*
Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets.
Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $T_1, \ldots, T_k \subseteq \mathbb{R}^d$ semialgebraic sets. Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^\mathbb{N}$ by

$$w_n = (A^n x \in T_1, \ldots, A^n x \in T_k).$$

Intuition: $w_n$ records to which sets $A^n x$ belongs to at each step $n$. 
MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k \subseteq \mathbb{R}^d$ semialgebraic sets. Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^\mathbb{N}$ by

$$w_n = (A^n x \in \mathcal{T}_1, \ldots, A^n x \in \mathcal{T}_k).$$

Intuition: $w_n$ records to which sets $A^n x$ belongs to at each step $n$.

Problem: given an MSO formula $\Psi$ over $(\mathbb{N}, <)$, decide whether $w \models \Psi$.

Examples: $P_i(n)$ means $A^n x \in \mathcal{T}_i$

- $\mathcal{T}_i$ is reachable: $\exists n. P_i(n)$
- whenever $\mathcal{T}_i$ is visited $\mathcal{T}_j$ is visited some point later:

$$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$
MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $T_1, \ldots, T_k \subseteq \mathbb{R}^d$ semialgebraic sets. Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^\mathbb{N}$ by

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Examples: $P_i(n)$ means $A^n x \in T_i$

- $T_i$ is reachable: $\exists n. P_i(n)$
- whenever $T_i$ is visited $T_j$ is visited some point later:
  $$\forall n : P_i(n) \Rightarrow (\exists m > n : P_j(m))$$
- in target $T_i$ at every odd position:
  $$\exists O \subseteq \mathbb{N} : \text{formula to define odd numbers} \land \forall x : x \in O \Rightarrow P_i(x)$$
MSO model-checking

Given $x \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{n \times n}$ and $T_1, \ldots, T_k \subseteq \mathbb{R}^d$ semialgebraic sets. Let $\Sigma = \{0, 1\}^k$ and define $w \in \Sigma^\mathbb{N}$ by

$$w_n = (A^nx \in T_1, \ldots, A^nx \in T_k).$$

Intuition: $w_n$ records to which sets $A^nx$ belongs to at each step $n$. Problem: given an MSO formula $\Psi$ over $(\mathbb{N}, <)$, decide whether $w \models \Psi$.

Theorem (Karimov, Lefaucheux, Ouaknine, Purser, Varonka, Whiteland, Worrell)

This is decidable if all $T_i$ either have intrinsic dimension 1 or are included in a subspace of dimension 3.

Examples: $P_i(n)$ means $A^nx \in T_i$

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- whenever $T_i$ is visited $T_j$ is visited some point later:

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Continuous linear dynamical systems

\[ x'(t) = Ax(t) \]

\[ x(0) = x_0 \]
Continuous linear dynamical systems

Linear differential equation:

\[ x'(t) = Ax(t) \quad x(0) = x_0 \]

Example:

\[ x'(t) = 7x(t) \]

\[ \sim x(t) = e^{7t} \]
Continuous linear dynamical systems

Linear differential equation:

\[ x'(t) = Ax(t) \quad x(0) = x_0 \]

Example:

\[ x'(t) = 7x(t) \]

\[ \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -x_1(t) \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ x(t) = e^{7t} \]

\[ \begin{cases} x_1(t) = \sin(t) \\ x_2(t) = \cos(t) \end{cases} \]
Continuous linear dynamical systems

Linear differential equation:

\[ x'(t) = Ax(t) \quad x(0) = x_0 \]

General solution form:

\[ x(t) = e^{At} x_0 \]

where \( e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} \)
### Continuous Skolem problem

Given $x$, $y$ and $A$, decide if $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$. 

---

Continuous reachability

Continuous positivity problem

Given $x$, $y$ and $A$, decide whether $x^T e^{At} y \geq 0$ for all $t \geq 0$. 

Continuous positivity is inter-reducible with continuous Skolem. 

The decidability of all these problems is also open!
Continuous reachability

Continuous Skolem problem

Given $x$, $y$ and $A$, decide if $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

Bounded continuous Skolem problem

Given $x$, $y$ and $A$, decide if $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$. 

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**Continuous positivity Problem**

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The decidability of all these problems is also open!
A link with number theory

Some reachability questions look like this:

$$\exists t \in \mathbb{R}. \ 42t^7 = 56 \land e^{3t} - e^t = 9$$
Some reachability questions look like this ($P, Q$ polynomials):

$$\exists t \in \mathbb{R}. P(t) = 0 \land Q(e^t) = 0$$

Claim: impossible except possibly for $t = 0$ (easy to check)

Algebraic number: root of polynomial with integer coefficients

Transcendental number: not algebraic, e.g. $e, \pi$

Theorem (Special case of Lindemann–Weierstrass)

If $t$ is a nonzero algebraic number then $e^t$ is transcendental.

$P(t)$ = 0 so $t$ is algebraic (by definition)

Lindemann–Weierstrass: $e^t$ transcendental (unless $t = 0$)

Hence $Q(e^t) \neq 0$ (except maybe if $t = 0$)
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**Theorem (Special case of Lindemann–Weierstrass)**

*If $t$ is a nonzero algebraic number then $e^t$ is transcendental.*

- $P(t) = 0$ so $t$ is algebraic (by definition)
- Lindemann–Weierstrass: $e^t$ transcendental (unless $t = 0$)
- hence $Q(e^t) \neq 0$ (except maybe if $t = 0$)
In general,

\[ x^T e^{At} y = \sum_{i=1}^{d} P_i(t) e^{\lambda_i t} \]

where \( P_i \) polynomial, \( \lambda_i \in \mathbb{C} \) eigenvalues of \( A \).
Exponential polynomial

In general,

\[ x^T e^{At} y = \sum_{i=1}^{d} P_i(t) e^{\lambda_i t} \]

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Lindemann–Weierstrass’s theorem is not enough to solve the continuous Skolem problem.
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**Theorem (Wilkie and MacIntyre)**

*If Schanuel’s conjecture is true, then, for each \( k \in \mathbb{N} \), the first-order theory of the structure \( (\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos \upharpoonright_{[0,k]}, \sin \upharpoonright_{[0,k]}) \) is decidable.*

- algorithm always correct, only termination requires the conjecture
Exponential polynomial

In general,

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- algorithm always correct, only termination requires the conjecture
- this makes many problem (inc. continuous Skolem) decidable!

**What is Schanuel’s conjecture?**
Schanuel’s conjecture

If \( z_1, \ldots, z_n \) that are \textbf{linearly independent} over \( \mathbb{Q} \), then at least \( n \) numbers among \( z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n} \) are \textbf{algebraically independent}.

Example:
\[
\begin{align*}
z_1 &= i\pi, \\
z_2 &= 1; \\
e^{z_1} &= -1, \\
e^{z_2} &= e.
\end{align*}
\]

Clearly \( z_1 \) and \( z_2 \) are linearly independent over \( \mathbb{Q} \). So at least 2 of \( i\pi, 1, -1, e \) are algebraically independent. But 1 is algebraic so \( \pi \) and \( e \) are algebraically independent.

Summary:
- Schanuel implies that \( \pi, e, \pi+e, e\pi, \ldots \) are transcendental.
- \( \pi \) and \( e \) are known to be transcendental.
- \( \pi+e \) is not known to be transcendental.
Schanuel’s conjecture

If $z_1, \ldots, z_n$ that are \textit{linearly independent} over $\mathbb{Q}$, then at least $n$ numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are \textit{algebraically independent}.

Example: $\pi$ and $e$ are algebraically independent

$$z_1 = i\pi, z_2 = 1 \leadsto e^{z_1} = -1, e^{z_2} = e.$$
Schanuel’s conjecture

If $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$, then at least $n$ numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are algebraically independent.

Example: $\pi$ and $e$ are algebraically independent

$$z_1 = i\pi, \quad z_2 = 1 \quad \sim \quad e^{z_1} = -1, \quad e^{z_2} = e.$$ 

Clearly $z_1$ and $z_2$ are linearly independent over $\mathbb{Q}$. So at least 2 of $i\pi, 1, -1, e$ are algebraically independent. But 1 is algebraic so $\pi$ and $e$ are algebraically independent.
Schanuel’s conjecture

If $z_1, \ldots, z_n$ that are linearly independent over $\mathbb{Q}$, then at least $n$ numbers among $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ are algebraically independent.

Example: $\pi$ and $e$ are algebraically independent

$$z_1 = i\pi, z_2 = 1 \implies e^{z_1} = -1, e^{z_2} = e.$$  

Clearly $z_1$ and $z_2$ are linearly independent over $\mathbb{Q}$. So at least 2 of $i\pi, 1, -1, e$ are algebraically independent. But 1 is algebraic so $\pi$ and $e$ are algebraically independent.

Summary:

- Schanuel implies that $\pi$, $e$, $\pi + e$, $e\pi$, ... are transcendental.
- $\pi$ and $e$ are known to be transcendental
- $\pi + e$ is not known to be transcendental
Continuous reachability

Bounded continuous Skolem problem: given $x, y$ and $A$, decide if
- **unbounded:** $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0$.
- **bounded:** $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0$.

**Theorem (Chonev, Ouaknine and Worrell, 2016)**

*The bounded continuous Skolem Problem is decidable subject to Schanuel’s conjecture.*
Continuous reachability

Bounded continuous Skolem problem: given $x$, $y$, and $A$, decide if

- **unbounded**: $\exists t \in [0, 1]$ such that $x^T e^{At} y = 0.$
- **bounded**: $\exists t \in \mathbb{R}$ such that $x^T e^{At} y = 0.$

Theorem (Chonev, Ouaknine and Worrell, 2016)

The bounded continuous Skolem Problem is decidable subject to Schanuel’s conjecture.

Theorem (Chonev, Ouaknine and Worrell, 2016)

If the (unbounded) continuous Skolem Problem is decidable then the Diophantine-approximation types of all real algebraic numbers is computable.

In other words: it requires new mathematics...
More complicated programs

Linear loop with if

\[
\begin{align*}
x &:= 2^{-10} \\
y &:= 1 \\
\text{while } y \geq x \text{ do} & \\
\quad \text{if } y \geq 2x \text{ then} & \\
\quad \quad x, y &:= \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\quad \text{else} & \\
\quad \quad x, y &:= \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]
More complicated programs

Linear loop with if

\[ x := 2^{-10} \]
\[ y := 1 \]
while \( y \geq x \) do
  if \( y \geq 2x \) then
    \[
    \begin{bmatrix}
      x \\
      y
    \end{bmatrix} := \begin{bmatrix}
      2 & 0 \\
      1 & 4
    \end{bmatrix} \begin{bmatrix}
      x \\
      y
    \end{bmatrix}
    \]
  else
    \[
    \begin{bmatrix}
      x \\
      y
    \end{bmatrix} := \begin{bmatrix}
      2 & 3 \\
      -3 & 7
    \end{bmatrix} \begin{bmatrix}
      x \\
      y
    \end{bmatrix}
    \]

Reachability is trivially undecidable by simulating two counter automata
More complicated programs

Linear loop with if

\[
x := 2^{-10} \\
y := 1 \\
\text{while } y \geq x \text{ do} \\
\quad \text{if } y \geq 2x \text{ then} \\
\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\quad \text{else} \\
\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Nondeterministic loop

\[
x := 2^{-10} \\
y := 1 \\
\text{while true do} \\
\quad \text{non deterministically do} \\
\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\quad \text{or} \\
\quad \quad \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Reachability is trivially undecidable by simulating two counter automata
More complicated programs

Linear loop with if

\[ x := 2^{-10} \]
\[ y := 1 \]
while \( y \geq x \) do
  if \( y \geq 2x \) then
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 0 \\
    1 & 4
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]
  else
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
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    2 & 3 \\
    -3 & 7
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]

Nondeterminic loop

\[ x := 2^{-10} \]
\[ y := 1 \]
while true do
  non deterministically do
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 0 \\
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    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]
  or
    \[
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    :=
    \begin{bmatrix}
    2 & 3 \\
    -3 & 7
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    y
    \end{bmatrix}
    \]

Reachability is trivially undecidable by simulating two counter automata

- Overapproximate behaviours
- Nondeterminic
Example: 2D robot

State: \( \vec{u} = (x_\theta, y_\theta, x, y) \)

Discretized actions:
- rotate arm by \( \psi \)
- change arm length by \( \delta \)

Linear transformations

Rotate arm by \( \psi \):
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\leftarrow
\begin{pmatrix}
  \cos \psi & -\sin \psi \\
  \sin \psi & \cos \psi
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]
\[
\begin{pmatrix}
  x_\theta \\
  y_\theta
\end{pmatrix}
\leftarrow
\begin{pmatrix}
  \cos \psi & -\sin \psi \\
  \sin \psi & \cos \psi
\end{pmatrix}
\begin{pmatrix}
  x_\theta \\
  y_\theta
\end{pmatrix}
\]

Change arm length by \( \delta \):
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\leftarrow
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
+ \delta
\begin{pmatrix}
  x_\theta \\
  y_\theta
\end{pmatrix}
\]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ?

Example: $\exists n \in \mathbb{N}$ such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$$
Matrix problems

Input: \( A, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n \in \mathbb{N} \) such that \( A^n = C \)?

✓ Decidable (PTIME)

Example: \( \exists n \in \mathbb{N} \) such that

\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}
\]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ? ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ?

Example: $\exists n, m \in \mathbb{N}$ such that
\[
\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} ?
\]
Matrix problems

Input: \( A, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n \in \mathbb{N} \) such that \( A^n = C \) ?

\( \checkmark \) Decidable (PTIME)

Input: \( A, B, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n, m \in \mathbb{N} \) such that \( A^n B^m = C \) ?

\( \checkmark \) Decidable

Example: \( \exists n, m \in \mathbb{N} \) such that
\[
\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix} \]

?
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ? ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ? ✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^{k} A_i^{n_i} = C$ ?

Example: $\exists n, m, p \in \mathbb{N}$ such that

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}^m \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}^p = \begin{bmatrix} 81 & 260 \\ 0 & 1 \end{bmatrix}$$ ?
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ?
✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ?
✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^{k} A_i^{n_i} = C$ ?
✓ Decidable if $A_i$ commute × Undecidable in general

Example: $\exists n, m, p \in \mathbb{N}$ such that

\[
\begin{bmatrix}
2 & 3 \\
0 & 1 \\
\end{bmatrix}^n 
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
0 & 1 \\
\end{bmatrix}^m 
\begin{bmatrix}
2 & 5 \\
0 & 1 \\
\end{bmatrix}^p 
= 
\begin{bmatrix}
81 & 260 \\
0 & 1 \\
\end{bmatrix}
?
\]
Matrix problems

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n \in \mathbb{N}$ such that $A^n = C$ ? ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n, m \in \mathbb{N}$ such that $A^n B^m = C$ ? ✓ Decidable

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists n_1, \ldots, n_k \in \mathbb{N}$ such that $\prod_{i=1}^k A_i^{n_i} = C$ ? ✓ Decidable if $A_i$ commute × Undecidable in general

Input: $A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $C \in \langle \text{semigroup generated by } A_1, \ldots, A_k \rangle$ ?

Semigroup: $\langle A_1, \ldots, A_k \rangle = \text{all finite products of } A_1, \ldots, A_k$
Examples:

$$A_1 A_3 A_2 \quad A_1 A_2 A_1 A_2 \quad A_3^8 A_2 A_1^3 A_3^{42}$$
Matrix problems

Input: \( A, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n \in \mathbb{N} \) such that \( A^n = C \)?
✓ Decidable (PTIME)

Input: \( A, B, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n, m \in \mathbb{N} \) such that \( A^n B^m = C \)?
✓ Decidable

Input: \( A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists n_1, \ldots, n_k \in \mathbb{N} \) such that \( \prod_{i=1}^{k} A_i^{n_i} = C \)?
✓ Decidable if \( A_i \) commute  × Undecidable in general

Input: \( A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d} \) matrices
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Semigroup: \( \langle A_1, \ldots, A_k \rangle = \) all finite products of \( A_1, \ldots, A_k \)
Examples:
\[
A_1 A_3 A_2 \quad A_1 A_2 A_1 A_2 \quad A_3^8 A_2 A_1^3 A_3^{42}
\]
Every nontrivial extension of simple linear loops seems to lead to undecidable problems.
Discrete reachability problems

Every nontrivial extension of simple linear loops seems to lead to undecidable problems. What about the continuous setting?
RC circuit

\[ \dot{I} = 0 \]
\[ \dot{I}_R = -\frac{1}{RC} I \]
\[ \dot{V}_R = -\frac{1}{C} I \]
\[ \dot{Q} = I \]
\[ \dot{V}_C = \frac{1}{C} I \]

When the switch is closed:

\[ I = \frac{1}{R}(V - V_C) \]
\[ \dot{I}_R = -\frac{1}{RC} I \]
\[ \dot{V}_R = -\frac{1}{C} I \]

When the switch is open:

\[ I = \frac{1}{R}(V - V_C) \]
\[ \dot{I}_R = -\frac{1}{RC} I \]
\[ \dot{V}_R = -\frac{1}{C} I \]
RC circuit

OPEN

\[ \dot{I} = 0 \]
\[ \dot{I}_R = - \frac{1}{RC} I_R \]
\[ \dot{V}_R = - \frac{1}{C} I_R \]
\[ \dot{Q} = I_R \]
\[ \dot{V}_C = \frac{1}{C} I_R \]
RC circuit

OPEN

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{I}_R &= -\frac{1}{RC} I_R \\
\dot{V}_R &= -\frac{1}{C} I_R \\
Q &= I_R \\
\dot{V}_C &= \frac{1}{C} I_R
\end{align*}
\]

CLOSED

\[
\begin{align*}
\dot{I} &= -\frac{1}{RC} I_R \\
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OPEN

\[
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CLOSED

\[
\begin{align*}
\dot{I} &= \frac{1}{R}(V - V_C) \\
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\dot{V}_C &= \frac{1}{C}I_R
\end{align*}
\]
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates

Dynamics:
\[ e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} \]
Switching systems

Restricted hybrid system:
- linear dynamics
- no guards (nondeterministic)
- no discrete updates

Problem:
\[ e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} = C \]

What we control: \( t_1, t_2, t_3, t_4 \in \mathbb{R}_{\geq 0} \)
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$?

Example: $\exists t \in \mathbb{R}$ such that

$$\exp \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t \right) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$$
Related work in the continuous case

Input: \( A, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists t \in \mathbb{R} \) such that \( e^{At} = C \) ? ✓ Decidable (PTIME)

Example: \( \exists t \in \mathbb{R} \) such that

\[
\exp\left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t \right) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} ?
\]
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$ ? ✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At} e^{Bu} = C$ ?

Example: $\exists t, u \in \mathbb{R}$ such that

$$\exp\left(\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} t\right) \exp\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u\right) = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix}$$
Related work in the continuous case

Input: $A, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t \in \mathbb{R}$ such that $e^{At} = C$ ?
✓ Decidable (PTIME)

Input: $A, B, C \in \mathbb{Q}^{d \times d}$ matrices
Output: $\exists t, u \in \mathbb{N}$ such that $e^{At} e^{Bu} = C$ ?
× Unknown

Example: $\exists t, u \in \mathbb{R}$ such that
$$
\exp \left( \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} t \right) \exp \left( \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} u \right) = \begin{bmatrix} 1 & 60 \\ 0 & 1 \end{bmatrix}
$$ ?
Switching system

\[ x' = A_1 x \quad \rightarrow \quad x' = A_2 x \]

\[ x' = A_4 x \quad \leftarrow \quad x' = A_3 x \]

What about a loop?
Switching system

\[ x' = A_1 x \quad \rightarrow \quad x' = A_2 x \quad \rightarrow \quad x' = A_4 x \quad \leftarrow \quad x' = A_3 x \]

What about a loop?

Dynamics:

\[ e^{A_4 t_4'} e^{A_3 t_3'} e^{A_2 t_2'} e^{A_1 t_1'} e^{A_4 t_4} e^{A_3 t_3} e^{A_2 t_2} e^{A_1 t_1} \]
Switching system

\[ x' = A_1 x \quad \text{with} \quad x' = A_2 x \]

\[ x' = A_3 x \quad \text{with} \quad x' = A_4 x \]

Loop \iff clique

Remark:

zero time dynamics \((t_i = 0)\) are allowed
Switching system

\[ x' = A_1 x \quad \rightarrow \quad x' = A_2 x \quad \rightarrow \quad x' = A_4 x \quad \leftarrow \quad x' = A_3 x \]

Dynamics:

any finite product of \( e^{A_i t} \) \( \sim \) semigroup!
Switching system

\[ x' = A_1 x \quad \rightarrow \quad x' = A_2 x \]
\[ x' = A_4 x \quad \leftarrow \quad x' = A_3 x \]

Problem:

\[ C \in \mathcal{G} \quad ? \]

where

\[ \mathcal{G} = \langle \text{semigroup generated by } e^{A_i t} \text{ for all } t \geq 0 \rangle \]
Reachability for switching systems

Input: \( A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d} \) matrices
Output: \( \exists t_1, \ldots, t_k \geq 0 \) such that
\[
\prod_{i=1}^{n} e^{A_i t_i} = C
\]

Input: \( A_1, \ldots, A_k, C \in \mathbb{Q}^{d \times d} \) matrices
Output:

\( C \in \langle \text{semigroup generated by } e^{A_1 t}, \ldots, e^{A_k t} : t \geq 0 \rangle \)

Theorem (Ouaknine, P, Sous-Pinto, Worrell)

Both problems are:

- **Undecidable in general**
- **Decidable** when all the \( A_i \) commute
Some words about the proof (commuting case)

<table>
<thead>
<tr>
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<th>Semigroup Problem</th>
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<td>$\exists t_1, \ldots, t_k \geq 0 \text{ s.t. } \prod_{i=1}^{n} e^{A_i t_i} = C$?</td>
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reduce

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**reduce**

**Integer Linear Programming**

| $\exists n \in \mathbb{Z}^d \text{ s.t. } \pi B n \leq s$ | $s$ of the form: $a_0 + \log(a_1) + \cdots + \log(a_k)$ | $B, a_0, \ldots, a_k$ are algebraic |

How did we get from reals to integers with $\pi$?

$\alpha \iff t \in \log(\alpha) + 2\pi \mathbb{Z}$
Some words about the proof (commuting case)

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$\exists n \in \mathbb{Z}^d$ s.t. $\pi B n \leq s$

$s$ of the form:

$a_0 + \log(a_1) + \cdots + \log(a_k)$

$B, a_0, \ldots, a_k$ are algebraic

How did we get from reals to integers with $\pi$?

$$e^{it} = \alpha \iff t \in \log(\alpha) + 2\pi \mathbb{Z}$$
\[ \exists n \in \mathbb{Z}^d \text{ such that } \pi Bn \leq s \ ? \]

where \( s \) is a linear form in logarithms of algebraic numbers
There exists an integer vector $n \in \mathbb{Z}^d$ such that $\pi B n \leq s$ where $s$ is a linear form in logarithms of algebraic numbers.

Key ingredient: Diophantine approximations

- Finding integer points in cones: Kronecker’s theorem
Integer Linear Programming

\[ \exists n \in \mathbb{Z}^d \text{ such that } \pi Bn \leq s \quad ? \]

where \( s \) is a linear form in logarithms of algebraic numbers

Key ingredient: Diophantine approximations

- Finding integer points in cones: Kronecker’s theorem
- Compare linear forms in logs: Baker’s theorem

\[ \sqrt{2} + \log \sqrt{3} - 3 \log \sqrt{7} \quad \equiv \quad 1 + \log 9 - \log \sqrt[42]{666} \]
Some words about the proof (general case)

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<th>Hilbert’s Tenth Problem</th>
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<td>( \exists n \in \mathbb{Z}^d \text{ s.t. } p(n) = 0 )</td>
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Theorem (Matiyasevich)

_Hilbert’s Tenth Problem is undecidable_
Summary on reachability

Exact reachability is hard:
- Skolem/Positivity problem for linear loops (Open for 70 years)
- Every mild extension is undecidable
- Decidability requires very strong assumptions (commuting matrices)

Continuous vs discrete setting
- similar results
- different techniques
- continuous setting can leverage powerful results/conjectures