A truly universal ordinary differential equation

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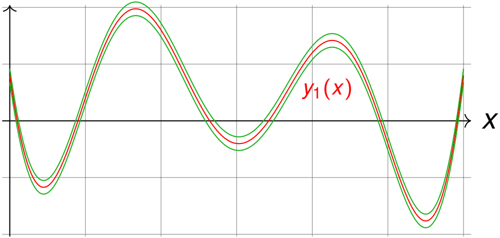
Universal differential algebraic equation (Rubel)



Theorem (Rubel, 1981)

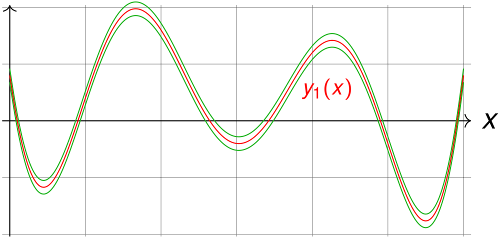
For any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$\begin{aligned} 3y'^4 y'' y''''^2 & -4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ & -12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 \qquad = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Universal differential algebraic equation (Rubel)



Theorem (Rubel, 1981)

There exists a **fixed** k and nontrivial polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

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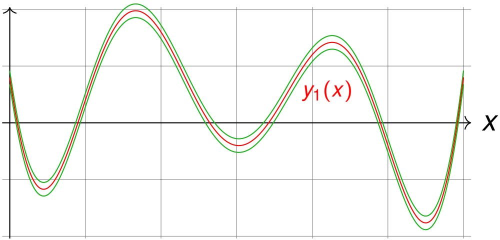

Open Problem

Can we have unicity of the solution with initial conditions ?

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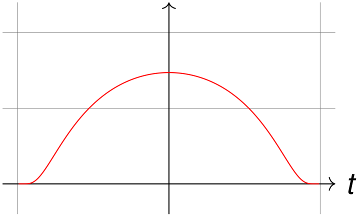
Rubel's ("disappointing") proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

  It satisfies $\quad (1-t^2)^2 f'(t) + 2tf(t) = 0.$



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- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at+b)$ satisfies

$$3y'^4y''y''''^2 \quad -4y'^4y''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y''''$$
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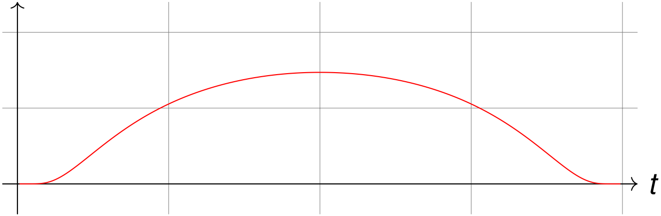

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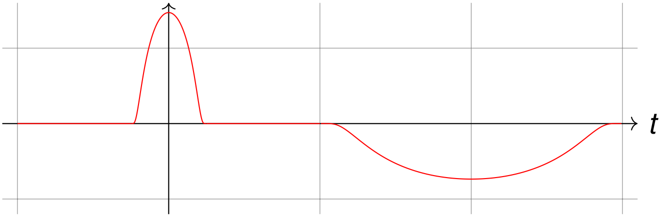
- Can glue together arbitrary many such pieces
  $\rightsquigarrow$crucial (and tricky) part of the proof



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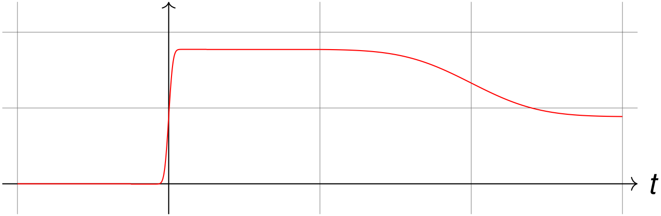
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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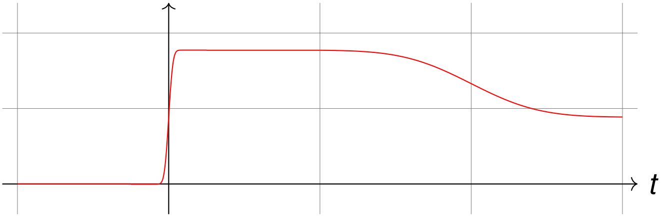

Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in C^0**

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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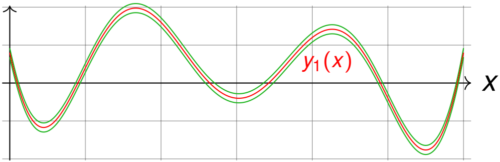
- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Universal explicit ordinary differential equation



Main result

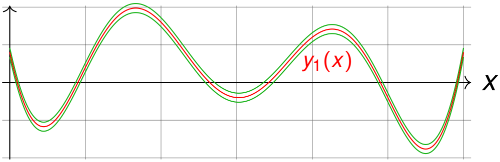
There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Universal explicit ordinary differential equation



Notes :

- **system** of ODEs,
- y must be analytic,
- we need $d \approx 300$.

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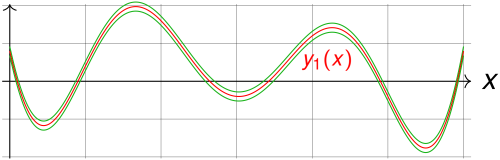

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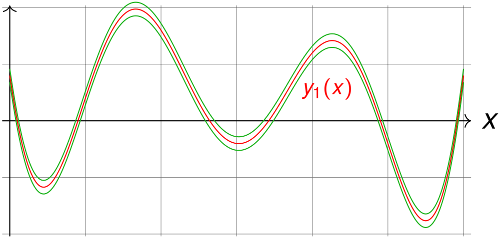
$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Futhermore, α is computable † from f and ε.

†. This statement can be made precise with the theory of Computable Analysis.

Universal DAE, again but better



Corollary of main result

There exists a **fixed** k and nontrivial polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

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has a **unique analytic solution** $y : \mathbb{R} \to \mathbb{R}$ and $\forall t \in \mathbb{R}$,

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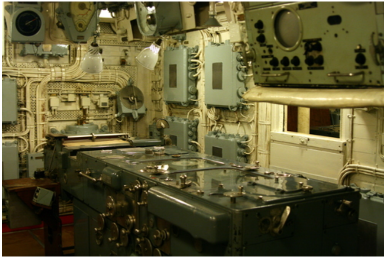
Some motivation

Polynomial ODEs correspond to analog computers :

Differential Analyser



British Navy mecanical computer

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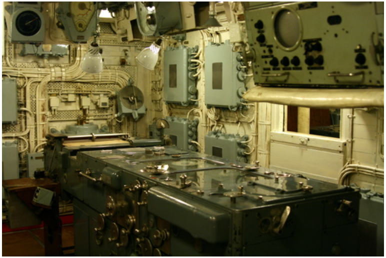
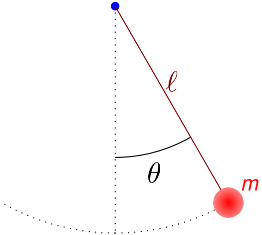


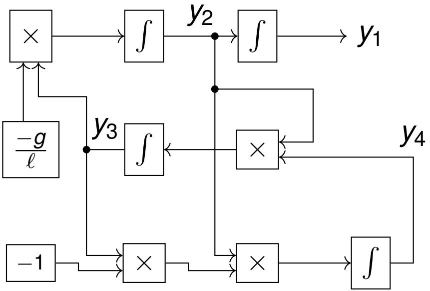
British Navy mecanical computer

- They are **equivalent** to Turing machines !
- One can **characterize P** with pODEs (ICALP 2016)

Take away : polynomial ODEs are a natural programming language.

Example of differential equation





General Purpose Analog Computer (GPAC)
Shannon's model of the Differential Analyser

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{\ell} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

A brief stop

Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by programming with ODEs.

Generable functions (total, univariate)

Definition

$f:\mathbb{R}\to\mathbb{R}$ is generable if there exists d, p and y_0 such that the solution y to

$$y(0)=y_0, \qquad y'(x)=p(y(x))$$

satisfies $f(x)=y_1(x)$ for all $x\in\mathbb{R}$.

Types

- $d\in\mathbb{N}$: dimension
- $\mathbb{Q}\subseteq\mathbb{K}\subseteq\mathbb{R}$: field
- $p\in\mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in $\mathbb{K}$)
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Note : existence and unicity of y by Cauchy-Lipschitz theorem.

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Example : $f(x)=x$ ▶ identity

$$y(0)=0,\qquad y'=1\quad\rightsquigarrow\quad y(x)=x$$

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Example : $f(x) = x^2$ ▶ squaring

$$\begin{array}{llll} y_1(0) = 0, & y_1' = 2y_2 & \rightsquigarrow & y_1(x) = x^2 \\ y_2(0) = 0, & y_2' = 1 & \rightsquigarrow & y_2(x) = x \end{array}$$

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Example : $f(x)=x^n$ ▶ n^{th} power

$$\begin{array}{llcl}
y_1(0)=0, & y_1'=ny_2 & \rightsquigarrow & y_1(x)=x^n \\
y_2(0)=0, & y_2'=(n-1)y_3 & \rightsquigarrow & y_2(x)=x^{n-1} \\
\dots & \dots & & \dots \\
y_n(0)=0, & y_n=1 & \rightsquigarrow & y_n(x)=x
\end{array}$$

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Example : $f(x) = \exp(x)$ ▶ exponential

$$y(0) = 1, \qquad y' = y \quad \rightsquigarrow \quad y(x) = \exp(x)$$

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Example : $f(x) = \sin(x)$ or $f(x) = \cos(x)$ ▶ sine/cosine

$$\begin{aligned} y_1(0) &= 0, & y_1' &= y_2 & \rightsquigarrow \quad y_1(x) &= \sin(x) \\ y_2(0) &= 1, & y_2' &= -y_1 & \rightsquigarrow \quad y_2(x) &= \cos(x) \end{aligned}$$

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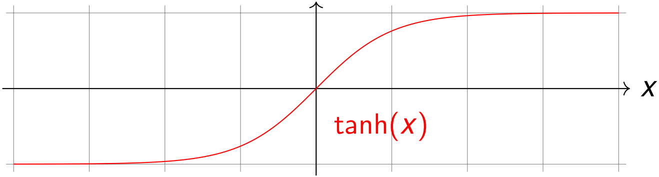
Example : $f(x)=\tanh(x)$ ▶ hyperbolic tangent

$$y(0)=0,\qquad y'=1-y^2\quad\rightsquigarrow\quad y(x)=\tanh(x)$$



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Example : $f(x)=\frac{1}{1+x^2}$ ▶ rational function

$$f'(x)=\frac{-2x}{(1+x^2)^2}=-2xf(x)^2$$

$$\begin{array}{lll} y_1(0)=1, & y_1'=-2y_2y_1^2 & \rightsquigarrow\quad y_1(x)=\frac{1}{1+x^2}\\ y_2(0)=0, & y_2'=1 & \rightsquigarrow\quad y_2(x)=x \end{array}$$

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Example : $f=g\pm h$ ▶ sum/difference

$$(g\pm h)'=g'\pm h'$$

assume :

$$z(0)=z_0,\qquad z'=p(z)\qquad\rightsquigarrow\qquad z_1=g$$
$$w(0)=w_0,\qquad w'=q(w)\qquad\rightsquigarrow\qquad w_1=h$$

then :

$$y(0)=z_{0,1}+w_{0,1},\qquad y'=p_1(z)\pm q_1(w)\qquad\rightsquigarrow\qquad y=z_1\pm w_1$$

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Example : $f = gh$ ▶ product

$$(gh)' = g'h + gh'$$

assume :

$$z(0) = z_0, \qquad z' = p(z) \qquad \rightsquigarrow \qquad z_1 = g$$
$$w(0) = w_0, \qquad w' = q(w) \qquad \rightsquigarrow \qquad w_1 = h$$

then :

$$y(0) = z_{0,1} w_{0,1}, \qquad y' = p_1(z) w_1 + z_1 q_1(w) \qquad \rightsquigarrow \qquad y = z_1 w_1$$

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Example : $f = \frac{1}{g}$ ▶ inverse

$$f' = \frac{-g'}{g^2} = -g' f^2$$

assume :

$$z(0) = z_0, \qquad z' = p(z) \qquad \rightsquigarrow \qquad z_1 = g$$

then :

$$y(0) = \frac{1}{z_{0,1}}, \qquad y' = -p_1(z) y^2 \qquad \rightsquigarrow \qquad y = \frac{1}{z_1}$$

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satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

Types

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Example : $f = \int g$ ▶ integral

assume :

$$z(0) = z_0, \qquad z' = p(z) \quad \rightsquigarrow \quad z_1 = g$$

then :

$$y(0) = 0, \qquad y' = z_1 \quad \rightsquigarrow \quad y = \int z_1$$

Generable functions (total, univariate)

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Example : $f=g'$ ▶ derivative

$$f'=g''=(p_1(z))'=\nabla p_1(z)\cdot z'$$

assume :

$$z(0)=z_0, \qquad z'=p(z) \qquad \rightsquigarrow \qquad z_1=g$$

then :

$$y(0)=p_1(z_0), \qquad y'=\nabla p_1(z)\cdot p(z) \qquad \rightsquigarrow \qquad y=z_1''$$

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Example : $f=g\circ h$ ▶ composition

$$(z\circ h)'=(z'\circ h)h'=p(z\circ h)h'$$

assume :

$$\begin{aligned} z(0)&=z_0, & z'&=p(z) &\rightsquigarrow\quad z_1&=g\\ w(0)&=w_0, & w'&=q(w) &\rightsquigarrow\quad w_1&=h \end{aligned}$$

then :

$$y(0)=z(w_0),\qquad y'=p(y)z_1\quad\rightsquigarrow\quad y=z\circ h$$

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Is this coefficient in $\mathbb{K}$?

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then :

$$y(0) = z(w_0), \qquad y' = p(y)z_1 \quad \rightsquigarrow \quad y = z \circ h$$

Is this coefficient in $\mathbb{K}$? Fields with this property are called generable.

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Example : $f' = \tanh \circ f$ ▶ Non-polynomial differential equation

$$f'' = (\tanh' \circ f)f' = (1 - (\tanh \circ f)^2)f'$$

$$\begin{array}{lll} y_1(0) = f(0), & y_1' = y_2 & \rightsquigarrow \quad y_1(x) = f(x) \\ y_2(0) = \tanh(f(0)), & y_2' = (1 - y_2^2)y_2 & \rightsquigarrow \quad y_2(x) = \tanh(f(x)) \end{array}$$

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Example : $f(0)=f_0, f'=g\circ f$ ▶ Initial Value Problem (IVP)

$$f'=g''=(p_1(z))'=\nabla p_1(z)\cdot z'$$

assume :

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Generable functions : a first summary

Nice theory for the class of total and univariate generable functions :

- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
- technicality on the field $\mathbb{K}$ of coefficients for stability under $\circ$
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Limitations :

- total functions
- univariate

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \qquad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

Types

- $n \in \mathbb{N}$: input dimension
- $d \in \mathbb{N}$: dimension
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$J_y(x) =$ Jacobian matrix of y at x

Notes :

- Partial differential equation !
- Unicity of solution y...
- ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3)$ ▶ monomial

$$y(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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$$\begin{array}{llll} y_1(0,0)=0, & \partial_{x_1} y_1 = y_3^2, & \partial_{x_2} y_1 = 3y_2y_3 & \rightsquigarrow \quad y_1(x) = x_1x_2^2 \\ y_2(0,0)=0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \rightsquigarrow \quad y_2(x) = x_1 \\ y_3(0,0)=0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \rightsquigarrow \quad y_3(x) = x_2 \end{array}$$

This is tedious !

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Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$ ▶ inverse function

$$y(1) = 1, \qquad \partial_x y = -y^2 \quad \rightsquigarrow \quad y(x) = \frac{1}{x}$$

Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$, ...
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
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Exercice : are all analytic functions generable ?

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Exercice : are all analytic functions generable ? No
Riemann Γ and ζ are not generable.

Why is this useful ?

Writing polynomial ODEs by hand is hard.

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Example (almost rounding function)

There exists a generable function round such that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $\lambda > 2$ and $\mu \geqslant 0$:

- if $x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant \frac{1}{2}$,
- if $x \in \left[n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant e^{-\mu}$.

Reminder of the result

Main result (reminder)

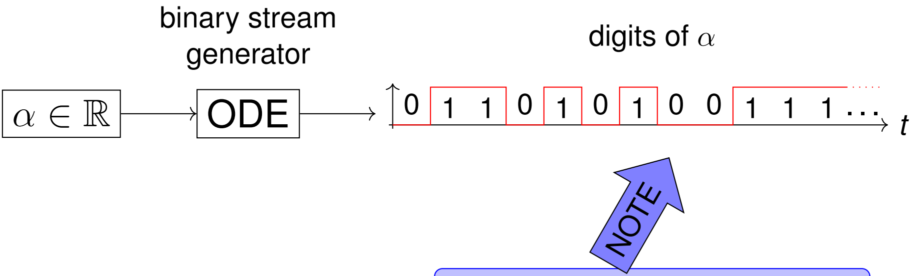
There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

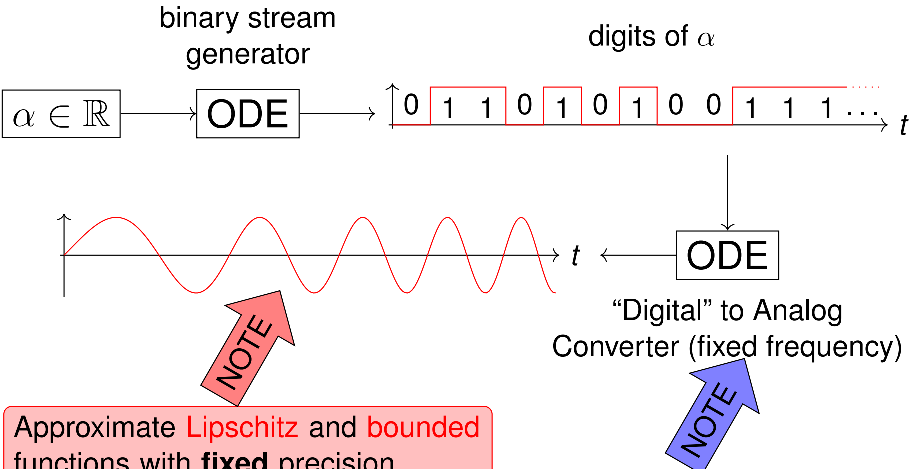
$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

A simplified proof



This is the ideal curve, the real one is an approximation of it.

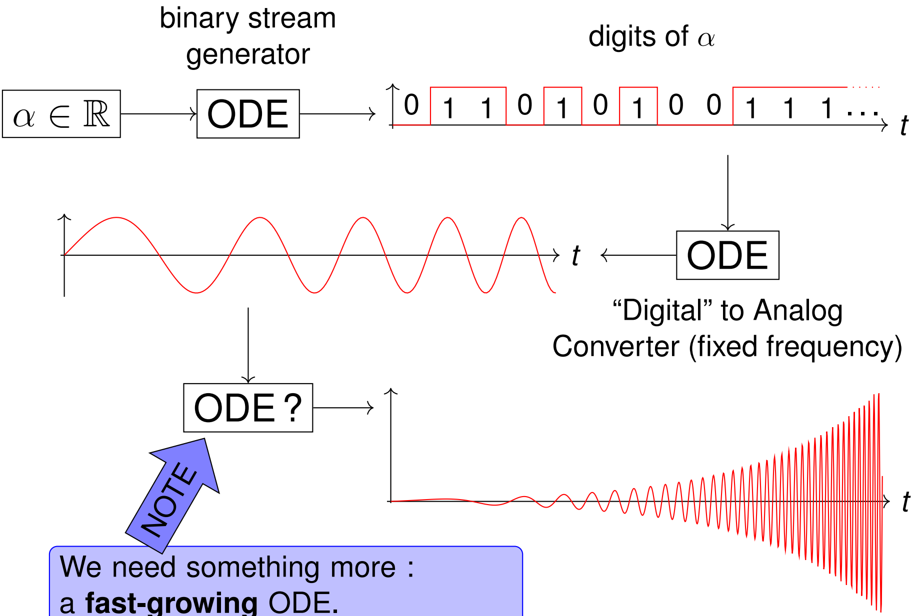
A simplified proof



Approximate Lipschitz and bounded functions with **fixed** precision.

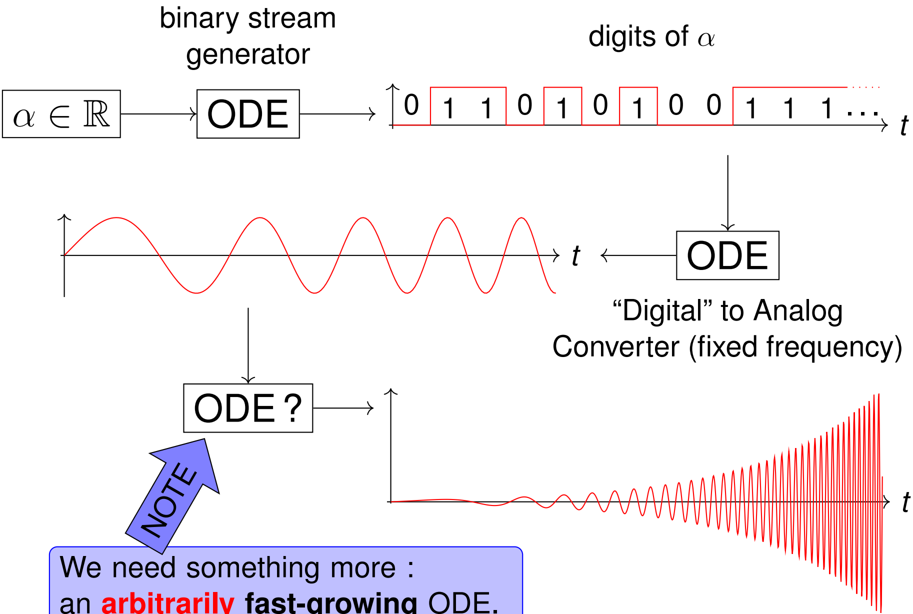
That's the trickiest part.

A simplified proof



We need something more :
a **fast-growing** ODE.

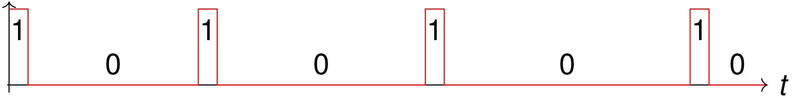
A simplified proof



We need something more :
an **arbitrarily** **fast-growing** ODE.

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2}\tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4,\lambda)} + 4\pi/3))$$

It's horrible, but generable

round is the mysterious rounding function...

A less simplified proof

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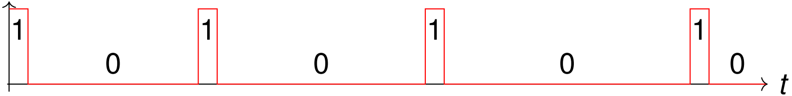
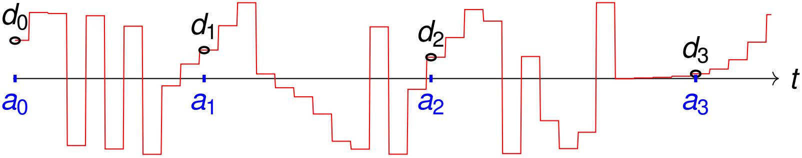


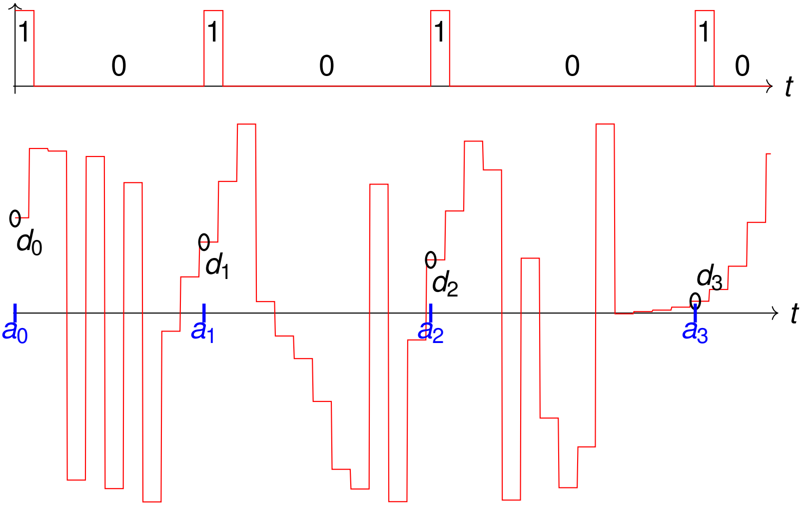


dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j<i} d_j$

$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4,\gamma)}))$$

round is the mysterious rounding function...

A less simplified proof



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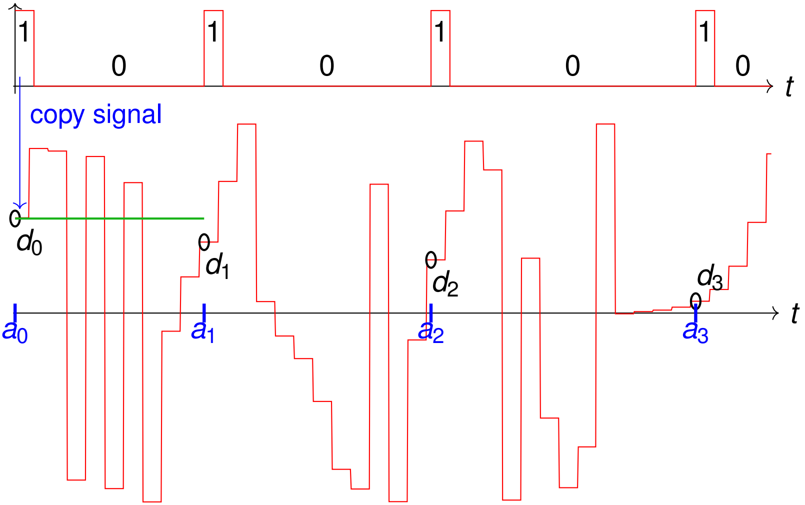

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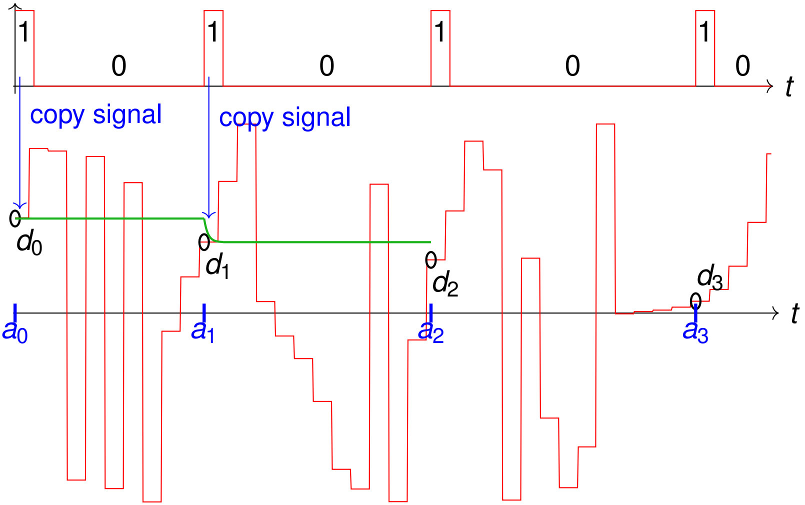

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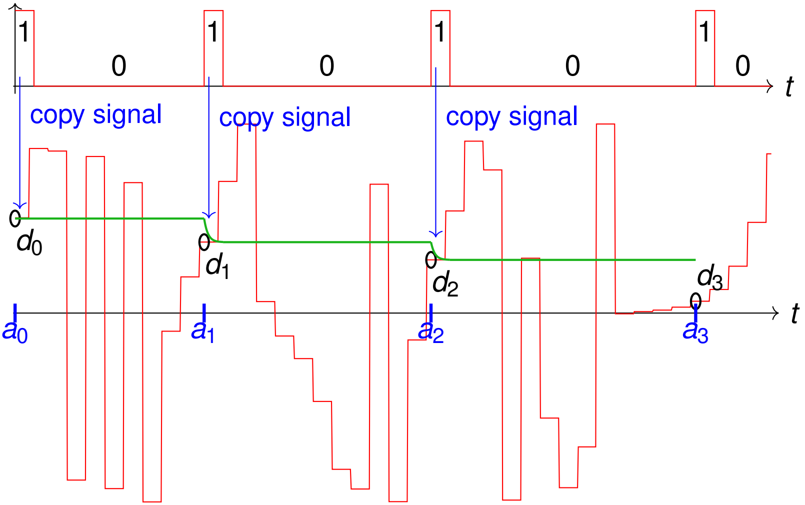

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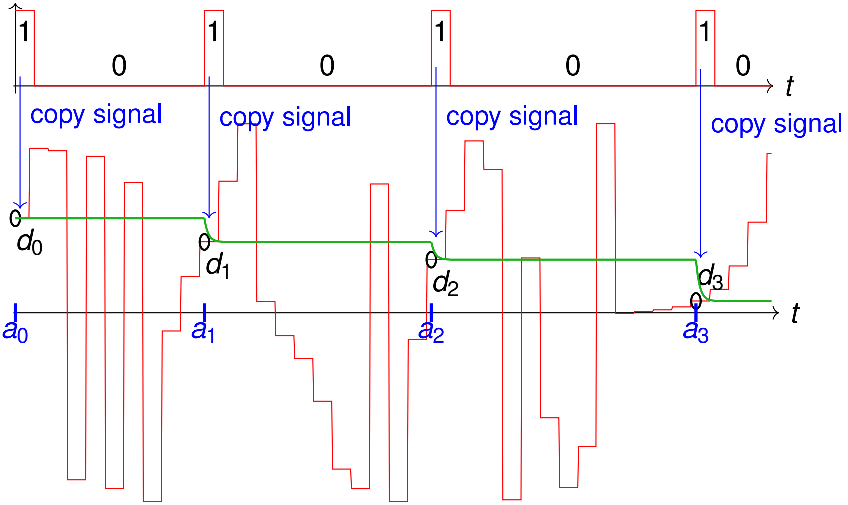

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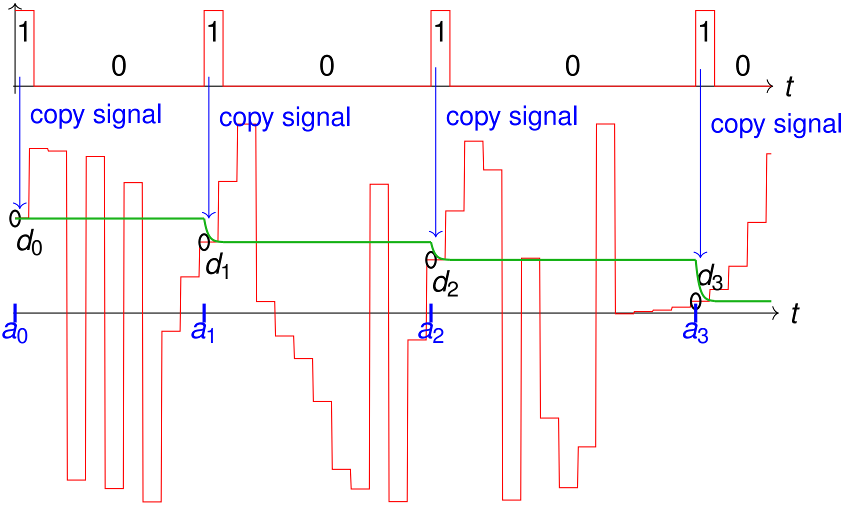
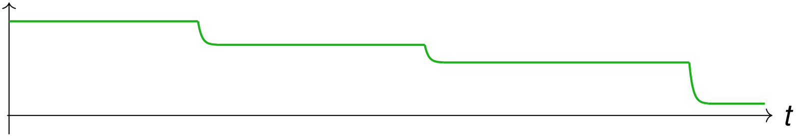


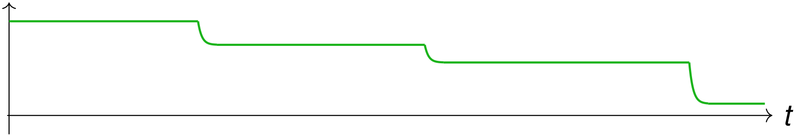
This copy operation is the “non-trivial” part.

A less simplified proof



We can do **almost piecewise constant functions...**

A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

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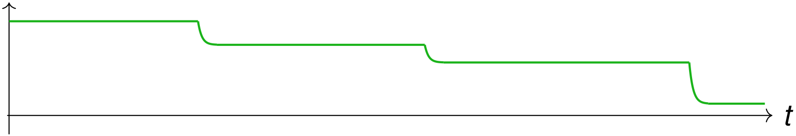

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How do we go to arbitrarily large and growing functions ? Can a polynomial ODE even have arbitrary growth ?

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \qquad \rightsquigarrow \qquad y_1(t) = \exp(t)$$

An old question on growth

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$$\begin{array}{lcl} y_1' = y_1 & \rightsquigarrow & y_1(t) = \exp(t) \\ y_2' = y_1 y_2 & \rightsquigarrow & y_1(t) = \exp(\exp(t)) \end{array}$$

An old question on growth

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&\dots & & \dots \\
y_n' &= y_1 \cdots y_n & \rightsquigarrow \quad & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t)
\end{aligned}$$

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Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

An old question on growth

$e_n(t) = \exp(\cdots \exp(t) \cdots)$ (n compositions)

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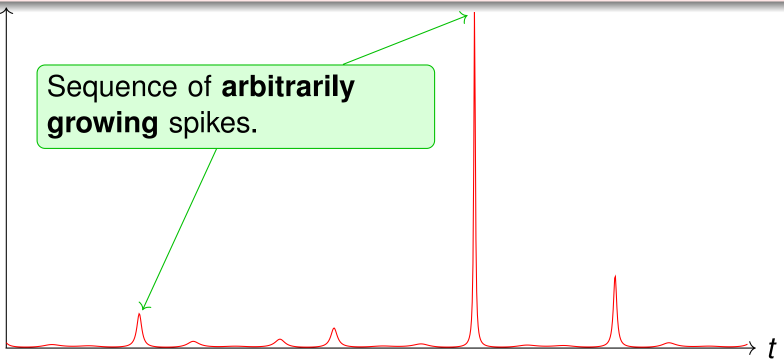
Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



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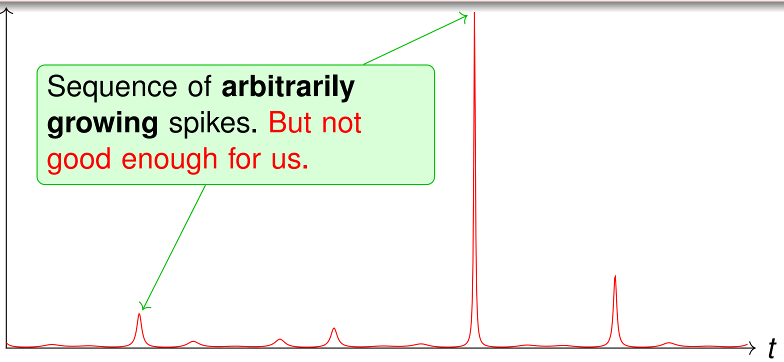

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Theorem (In the paper)

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

satisfies

$$y_1(t) \geqslant f(t), \qquad \forall t \geqslant 0.$$

An old question on growth

$e_n(t) = \exp(\cdots\exp(t)\cdots)$ (n compositions)

Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2-\cos(t)-\cos(\alpha t)}$$

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Note : both results require α to be **transcendental**. Conjecture still open for **rational** (or algebraic) coefficients.

Proof gem : iteration with differential equations

Assume f is generable, can we **iterate** f with an ODE ?
That is, build a generable y such that $y(x, n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$

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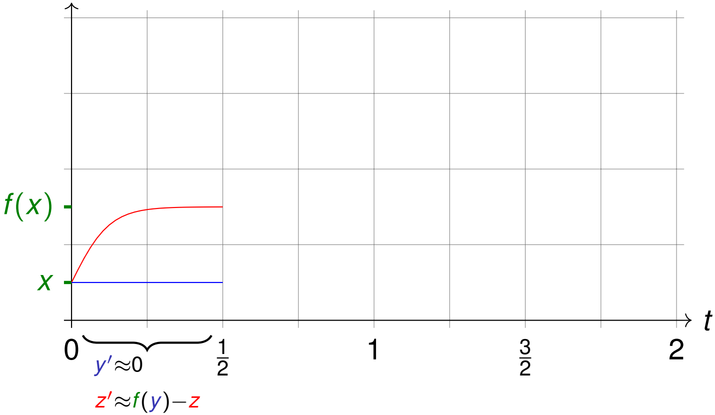

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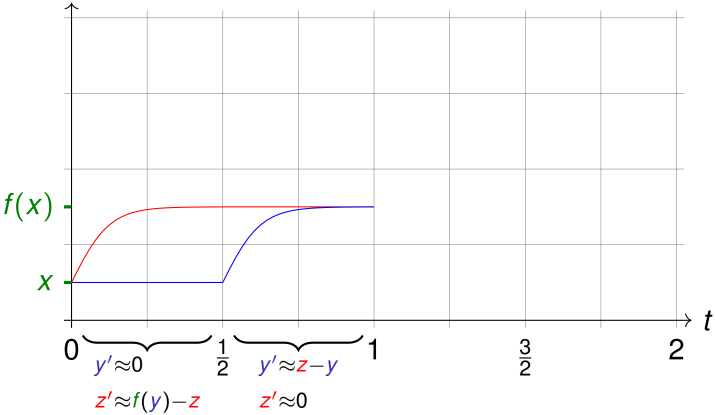

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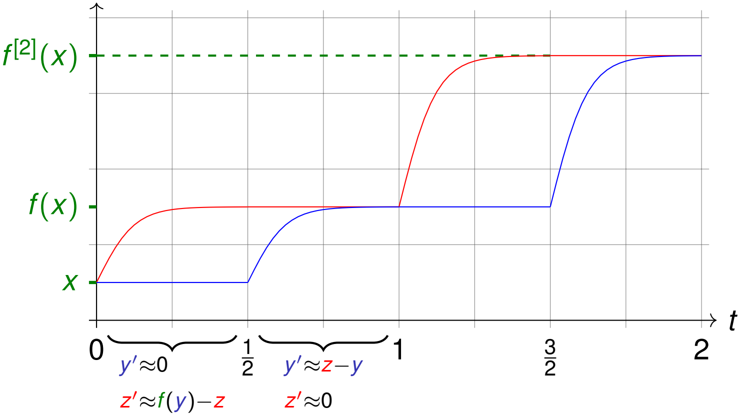


Main result, remark and end

Main result (reminder)

There exists a **fixed** (vector of) polynomial p such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

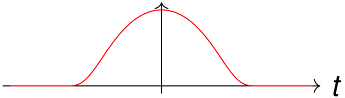
Futhermore, α is computable from f and ε.

Remarks :

- if f and ε are computable then α is computable
- if f or ε is **not computable** then α is **not computable**
- in all cases α is a horrible transcendental number

Non-unicity of solutions of DAEs even with conditions

Let $f(t) = \begin{cases} \exp(-\tan(t)^2) & \text{if } |t| < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$



Lemma : $f \in C^\infty(\mathbb{R})$ and for all $a, \lambda \in \mathbb{R}$, $g := t \mapsto \lambda f(a+t)$ satisfies

$$-2y'^6+6y''y'^4y-6y''^2y'^2y^2+31y'^4y^2+2y''^3y^3-8y''y'^2y^3+4y''^2y^4+16y'^2y^4=0. \qquad (1)$$

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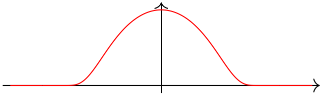

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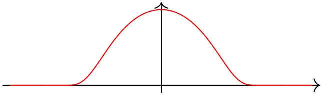

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A set of conditions for (1) is a collection of constraints of the form

$$y^{(k)}(a) = b \qquad \text{for some } k \in \mathbb{N} \text{ and } a, b \in \mathbb{R}.$$

Example : $y(0) = 1, y'(0) = 0, y''(42) = \pi$

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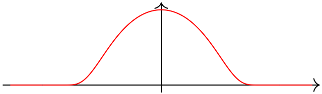

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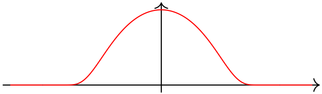

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Lemma : for any **finite** set of conditions, either (1) has no solution or at least one with bounded support.

Theorem : for any **finite** set of conditions, if (1) has a solution then it has infinitely many.