On the Computation of the Zariski Closure of Finitely Generated Groups of Matrices

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Motivation

Does this program halt?

Affine program

$x := 2^{-10}$
$y := 1$
while $y \geqslant x$ do

$$\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} 2 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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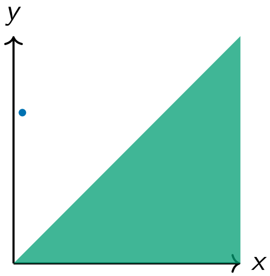

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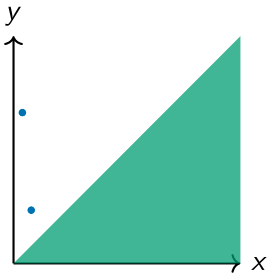

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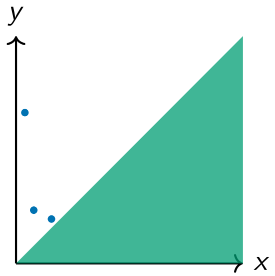

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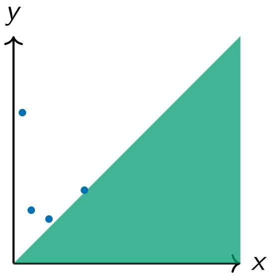

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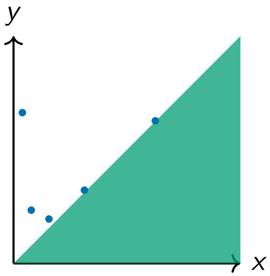

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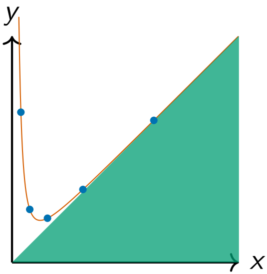

Certificate of non-termination:

$$x^2 y - x^3 = \frac{1023}{1073741824} \tag{1}$$

- (1) is an invariant: it holds at every step
- (1) implies the guard is true

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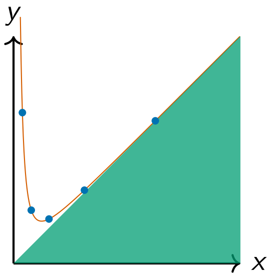

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$$x^2 y - x^3 = \frac{1023}{1073741824} \quad (1)$$

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Computing such invariants reduces to computing the **Zariski closure** of a semigroup of matrices.

Quantum automata

A matrix $U \in \mathbb{C}^{n \times n}$ is unitary if it is length preserving:

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A (measure once) quantum finite automaton (QFA):

- Σ: finite alphabet,
- $s \in \mathbb{C}^n$: vector of unit norm,
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Value of a word $w \in \Sigma^*$:

$$\mathrm{Val}_{\mathcal{A}}(w) = \|PX_w s\|_2^2 \qquad \text{where } X_w = X_{w_{|w|}} \cdots X_{w_1}$$

Interpretation: the probability of observing the quantum state in the acceptance space after having applied the operator sequence X_{w_1} to $X_{w_{|w|}}$ to the initial quantum states.

Quantum automata problems

Given a QFA $\mathcal{A}$ and a threshold λ:

Emptiness Problem

$\exists w \in \Sigma^*$ such that $\text{Val}_{\mathcal{A}}(w) \geq \lambda$?

Undecidable*: proof by reduction from PCP.

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Strict Emptiness Problem

$\exists w \in \Sigma^*$ such that $\mathsf{Val}_{\mathcal{A}}(w) > \lambda$?

Decidable*: reduces to computing the **Zariski closure** of a group of matrices

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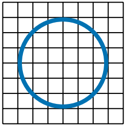
The Problem

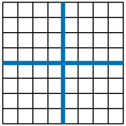
The Zariski topology

Algebraic set: set of common zeroes of a collection S of polynomials in $\mathbb{A}[x_1, \ldots, x_n]$:

$$V(S) = \{x \in \mathbb{A}^n : \forall p \in S, p(x) = 0\}$$



$x^2 + y^2 = c$



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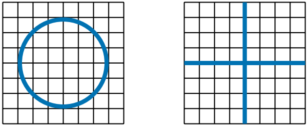

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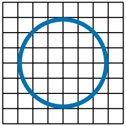

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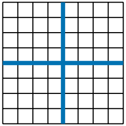

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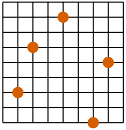
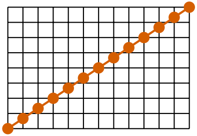
Zariski topology: closed sets are algebraic sets.

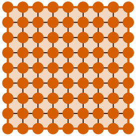
Zariski closure of a set X is the smallest algebraic set $\overline{X}$ that contains X.



$\overline{X} = X$ if X finite



$\overline{\mathbb{Z}} = \mathbb{A}$



$\overline{\mathbb{Z}}^2 = \mathbb{A}^2$

Zariski closure of finitely generated matrix semigroups

Given $A_1, \ldots, A_k \in \mathbb{A}^{n \times n}$, consider

$$\langle A_1, \ldots, A_k \rangle = \text{semigroup generated by the } A_i.$$

Problem: compute $\overline{\langle A_1, \ldots, A_k \rangle}$.

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Example:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \rightsquigarrow \qquad \langle S, T \rangle = \mathrm{SL}_2(\mathbb{Z})$$

then

$$\overline{\langle S, T \rangle} = SL_2(\overline{\mathbb{Z}}) = SL_2(\mathbb{A}) = \left\{ M \in \mathbb{A}^{n \times n} : \det(M) = 1 \right\}.$$

History of the problem

Given a finite set $S \subseteq \mathbb{A}^{n \times n}$ and $d \in \mathbb{N}$, define the "degree-d closure" as the smallest algebraic set that contains $\langle S \rangle$ and is defined by polynomials of total degree at most d.

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Theorem (Karr, 1974; Müller-Olm and Seidl, 2004)

There is an algorithm that computes, given S and d, the degree-d closure of $\langle S \rangle$, in time $O(|S| \cdot (n^2 + 1)^{3d})$.

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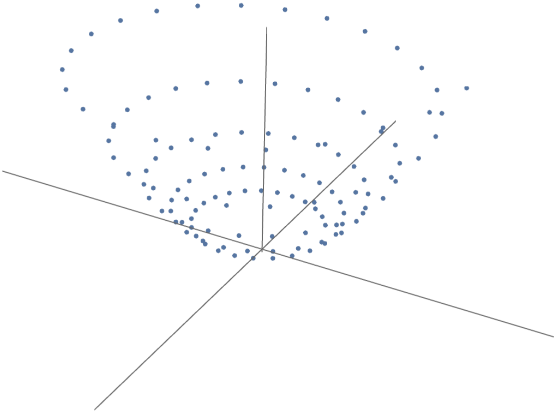
Remarks:

- most applications do **not** need the closure: a sufficiently good approximation is sufficient
- surely one can obtain an upper bound on d ?

Why fixed degree is not enough



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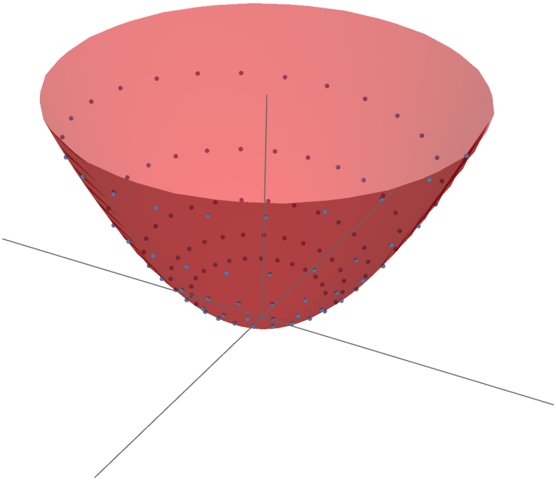

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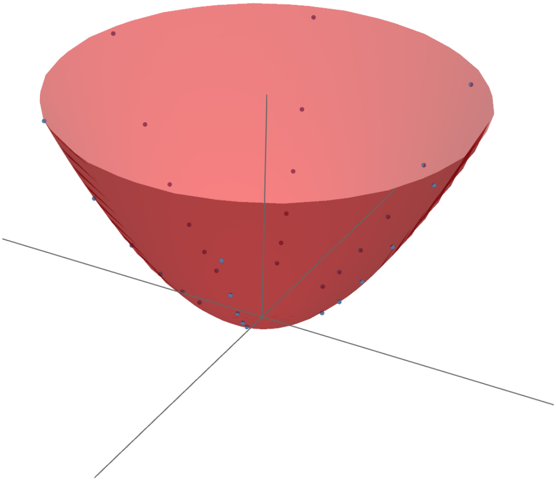

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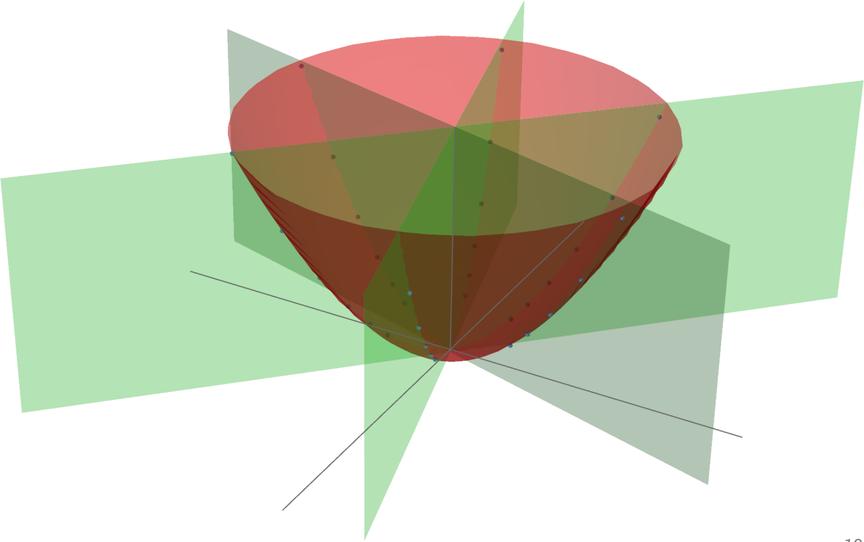

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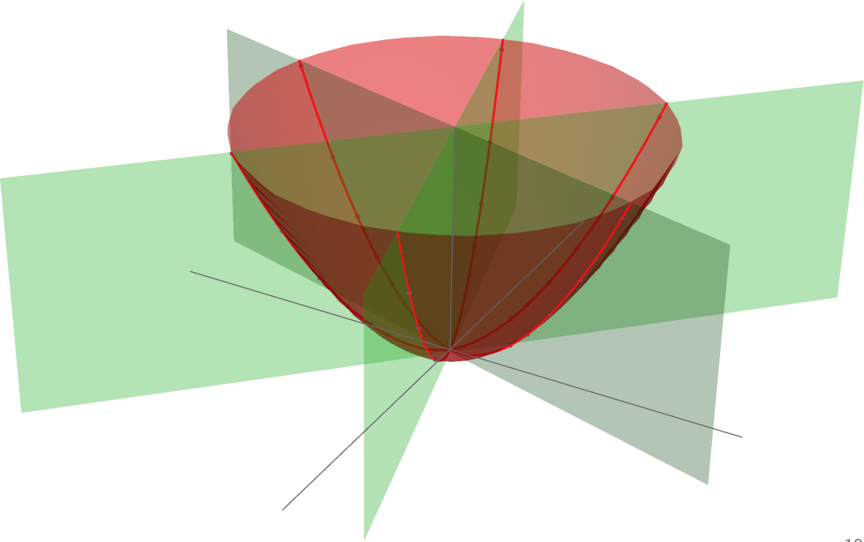

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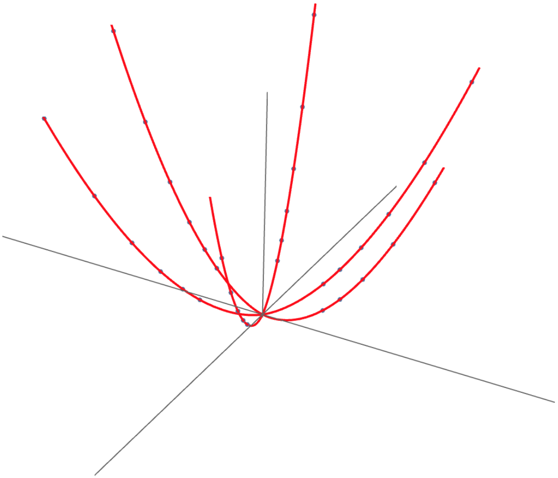

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Quantum automata and algebraic groups

Harm Derksen[a], Emmanuel Jeandel[b], Pascal Koiran[b,*]

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There is an algorithm that computes $\overline{\langle S \rangle}$ given a finite set S of matrices.

None of these algorithms puts a bound on the degree of the closure!

Main result

We obtain a degree bound for invertible matrices:

Theorem (ISSAC 2022)

Given a finite set S of invertible matrices of dimension n, the algebraic group $G := \overline{\langle S \rangle}$ can be defined with equations of degree at most septuply exponential in n.

Main result

We obtain a degree bound for invertible matrices:

Theorem (ISSAC 2022)

Let $n \in \mathbb{N}$ and let $S \subseteq \mathsf{GL}_n(\mathbb{Q})$ be a finite set of matrices whose entries have height at most h. Then the Zariski closure of the group generated by S can be represented by finitely many polynomials of degree at most $(\log h)^{2^{|S|^{\exp^4(\mathrm{poly}(n))}}}$ with coefficients in $\mathbb{Q}$, forming a basis of the vanishing ideal of the group generated by S. Furthermore, if G contains only semisimple elements then the degree can be bounded by $(\log h)^{2^{|S|^{2^{\mathrm{poly}(n)}}}}$.

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Corollary

The algebraic closure of a finitely generated matrix group is computable in elementary (octuply exponential) time.

Summary

Motivation:

- certifying non-termination of linear loops
- analysing quantum automata

Problem: compute the Zariski closure of a finitely generated group of matrices

- computable
- we obtained a septuly exponential bound on the degree of the closure

Future work:

- improve bound using ideas from differential Galois group algorithms
- study special classes of groups
- extend to semigroups

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$$\mathsf{GL}_n(\mathbb{A}) = \{(M, y) \in \mathbb{A}^{n^2+1} : \det(M) \cdot y = 1\}$$

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Key fact: if $S \subseteq \mathsf{GL}_n(\mathbb{A})$ then $\overline{\langle S \rangle}$ is an algebraic group

We analyse the structure of algebraic groups that come from finitely generated groups.

Can we hope for better ?

A closely related topic is the computation of the Galois group of a linear differential equation which is a linear algebraic group.

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We use many ideas from the above papers to prove our result.

Future work: use the techniques of Amzallag, Minchenko and Pogudin to reduce our bound

Remarks on lower bounds

A difficulty in the proof is that the degree bound must depend on the height of the entries:

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$$(2^p)^{1} \cdot (\tfrac{1}{2})^{p} = 1.$$

Therefore any polynomial that vanishes on $\langle A \rangle$ must also vanish on

$$\{\mathrm{diag}(x, y) : xy^p = 1\}$$

and thus be of degree at least $1 + p \geqslant \log(h)$.

Conclusion:

- even in dimension 2, the degree can be arbitrarily large and depends on the height.
- the exponential "lower bound" of Amzallag, Minchenko, Pogudin probably also works in our case

Chains of algebraic groups

The proof yields a potentially useful result on chains of algebraic groups:

Theorem

Let $n \in \mathbb{N}$, k be a number field, and $G_i = \overline{\langle S_i \rangle}$ for $S_i \subseteq \mathrm{GL}_n(k)$, $1 \leq i \leq \ell$, be such that $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_\ell$. Then

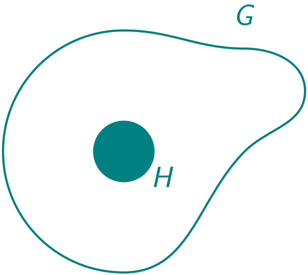
$$\ell \leq \exp\Big(\mathrm{poly}([k:\mathbb{Q}])\exp^3(\mathrm{poly}(n))\Big),$$

and $\ell \leq 2^{\mathrm{poly}(n[k:\mathbb{Q}])}$ if each G_i consists only of semisimple elements.

This may be useful to analyse the running time of algorithms.

The idea behind of our proof

G has a normal subgroup of finite index H:



Good properties

- we know $|G/H|$,
- we have degree bounds on H

The idea behind of our proof

G has a normal subgroup of finite index H: G is the union of $|G/H|$ copies of H

$\rightsquigarrow$ we can write equations for G from that of H and $|G/H|$.

Good properties

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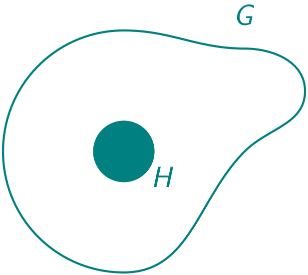
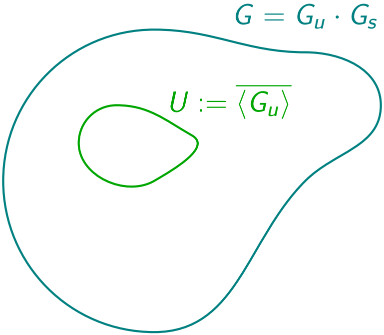


More details



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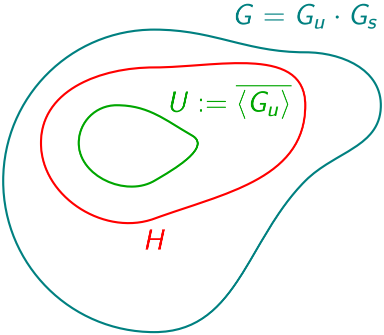

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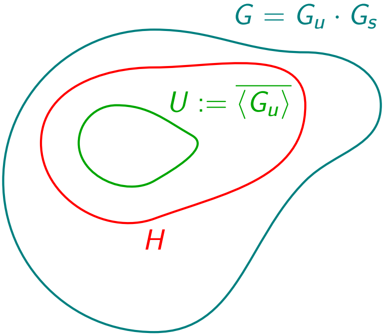

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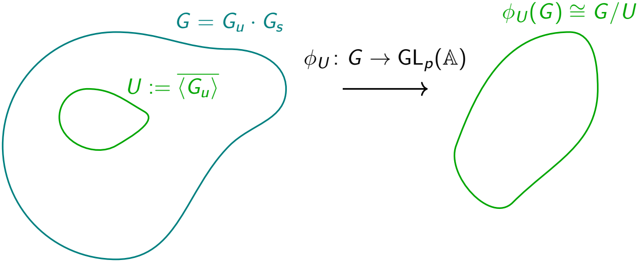
The quotient G/U is an algebraic group consisting only of semisimple elements.

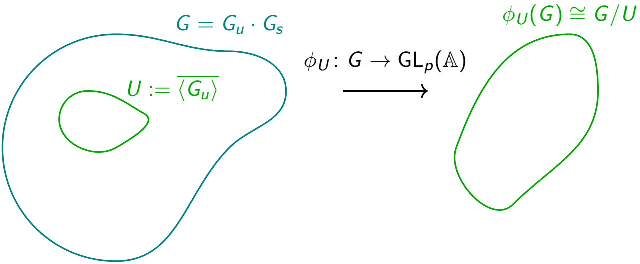
We can use this to reduce to the case of semisimple matrices!

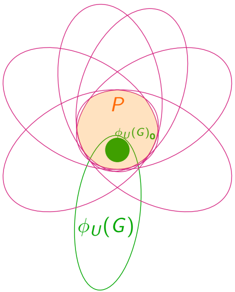
The construction



→ $U \lhd G$
→ $\phi_U(G) \cong G/U$ semisimple
→ Bound on U
→ Bound on the degree of equations defining ϕ_U **[Feng'15]**

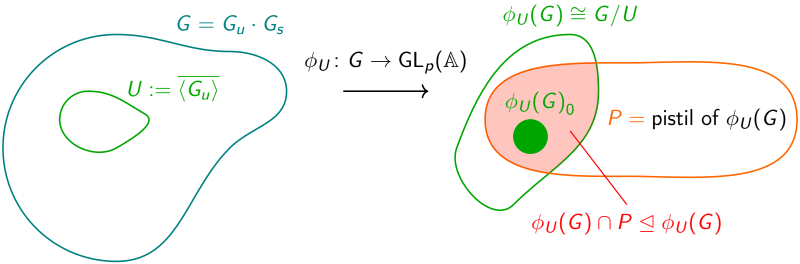
The construction

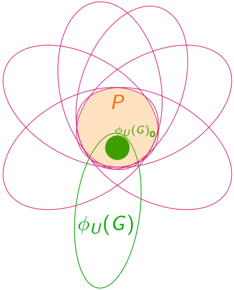




- $\to$ Pistil P of $\phi_U(G)$
- $\to$ $\phi_U(G) \triangleleft (\phi_U(G) \cap P) \triangleleft \phi_U(G)$
- $\to$ $\phi_U(G) \cap P$ finite index in $\phi_U(G)$
- $\to$ P commutative
- $\to$ P bounded by 1

The construction





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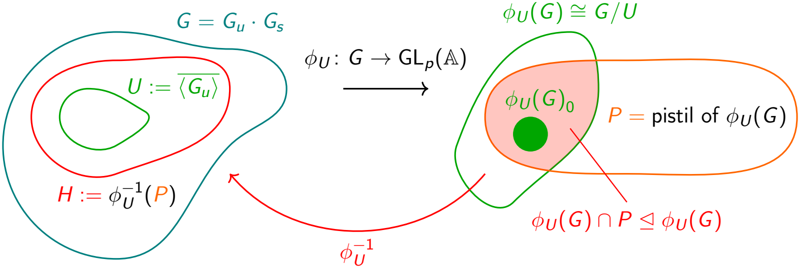

- → $U \lhd H \lhd G$
- → $\phi_U(G) \cong G/U$ semisimple
- → Bound on U
- → Bound on the degree of equations defining ϕ_U
- → Bound on H
- → H finite index in G
- → H/U commutative

Deciding Strict Emptiness for QFA

Strict Emptiness

Given a QFA $\mathcal{A}$ and a threshold λ:

$$\exists w \in \Sigma^* \text{ s.t. } \mathsf{Val}_{\mathcal{A}}(w) > \lambda \text{ ?}$$

Note: if such a word exists, it is a finite certificate. Enumerate!

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Observation:

(i) $\mathsf{Val}_{\mathcal{A}}(w) = f(X_w)$,

(ii) f is an Euclidean-continuous polynomial map.

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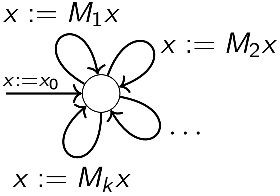
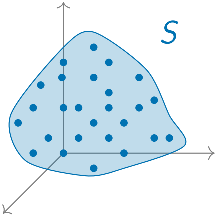
Crucial fact: the Euclidian closure of $\mathcal{X}$ is algebraic.

Finite certificate: $\overline{\mathcal{X}}$ can be finitely represented and is computable.
[Derksen et al.'05]

At the edge of decidability





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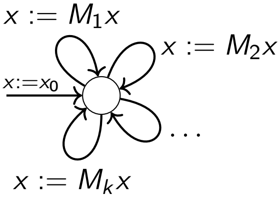

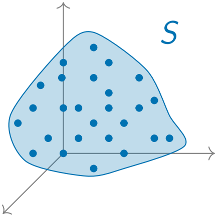

Theorem (Markov 1947)

There is a fixed finite set of 6×6 *integer matrices* S *such that the problem of deciding whether* $A \in \langle S \rangle$ *for a given* A *is undecidable.*

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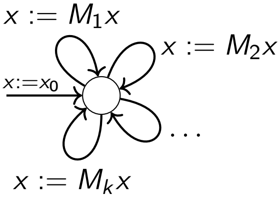

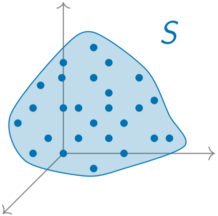

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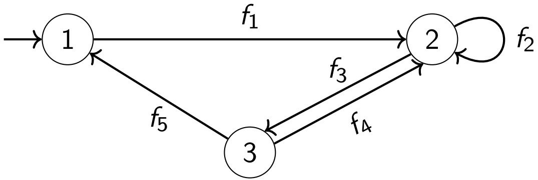
Theorem (Paterson 1970)

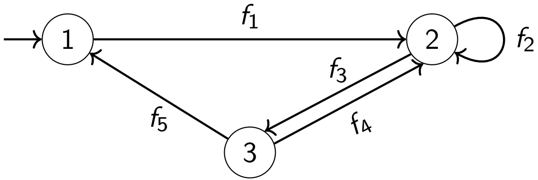
The problem of deciding, given $M_1, \ldots, M_k$*, whether* $0 \in \langle M_1, \ldots, M_k \rangle$ *is undecidable for* 3×3 *matrices.*

Affine programs



Affine programs

- Nondeterministic branching (no guards)



Affine programs

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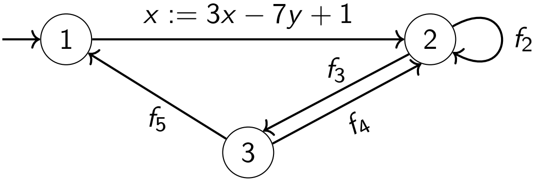

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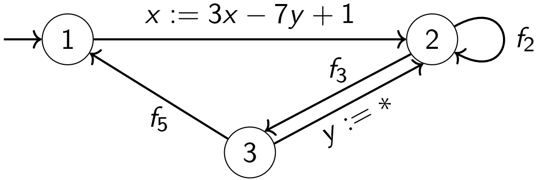

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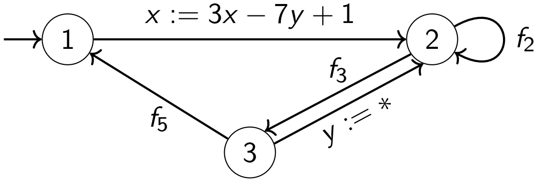

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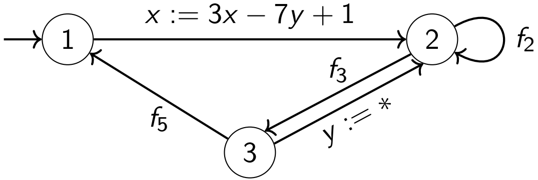
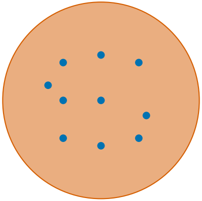


- Can overapproximate complex programs
- Covers existing formalisms:
  probabilistic, quantum, quantitative automata

Invariants

invariant = overapproximation of the reachable states



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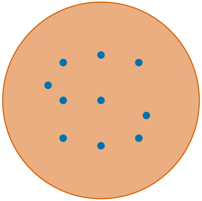
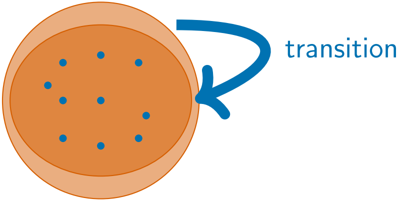
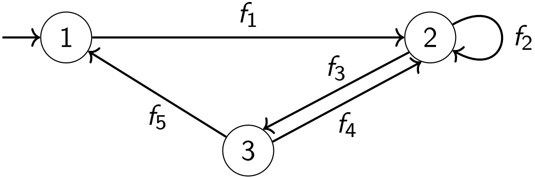


inductive invariant = invariant preserved by the transition relation

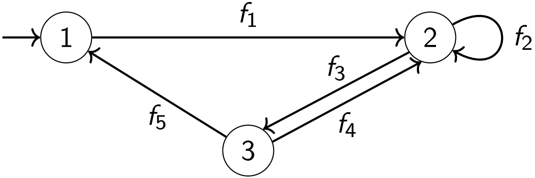


Inductive invariants: example



Inductive invariants: example

x, y, z range over $\mathbb{Q}$ $\quad$ $f_i : \mathbb{R}^3 \to \mathbb{R}^3$



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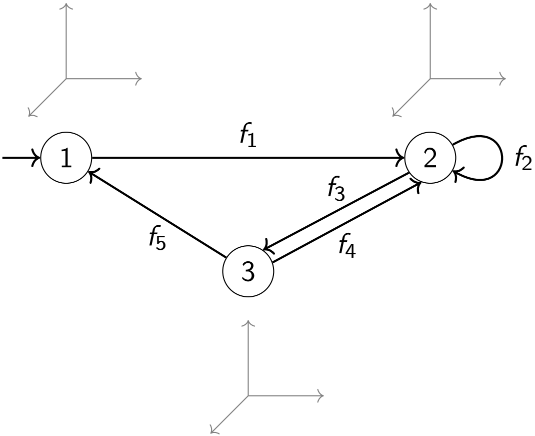

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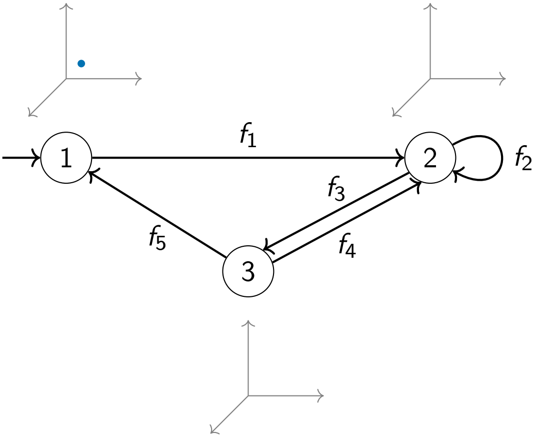

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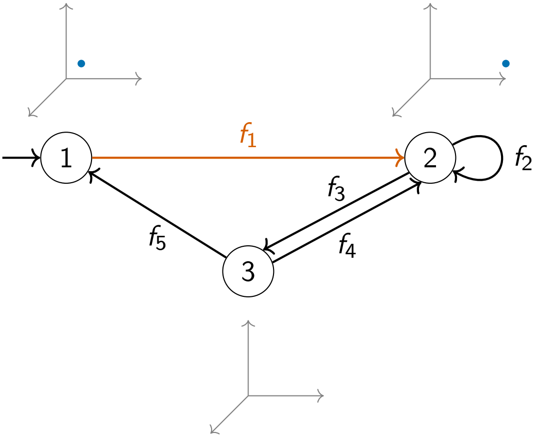

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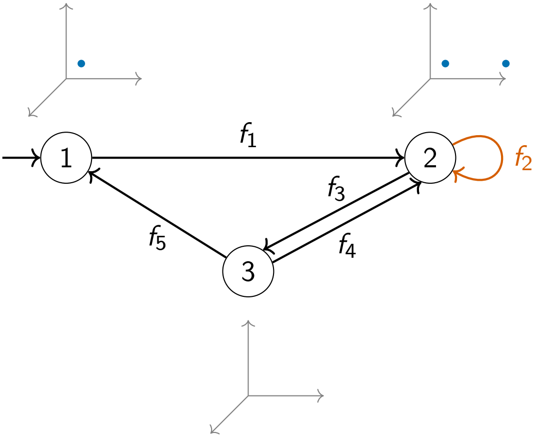

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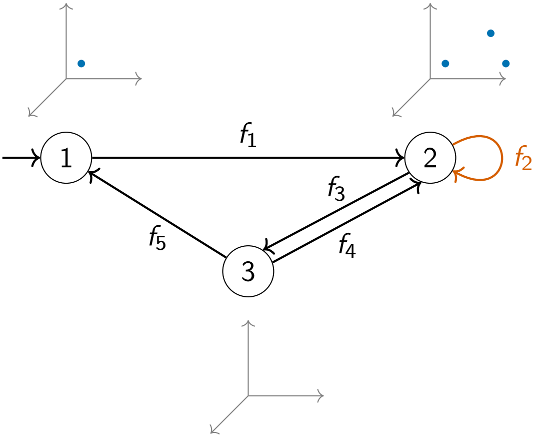

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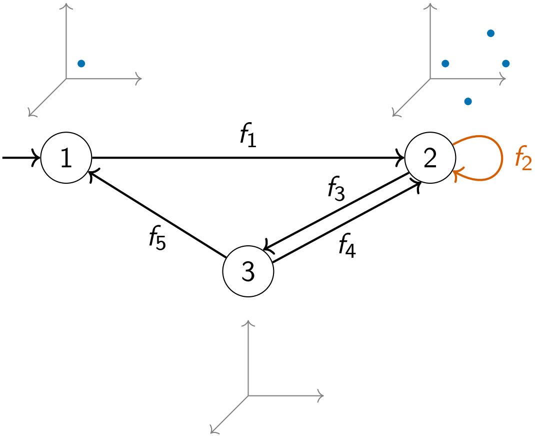

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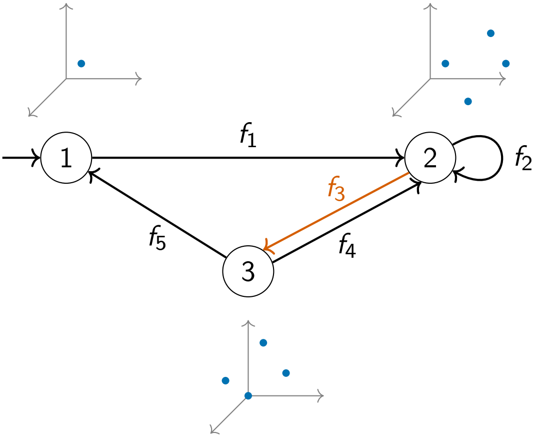

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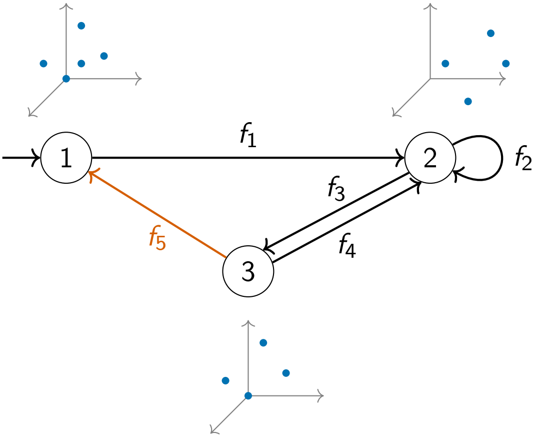

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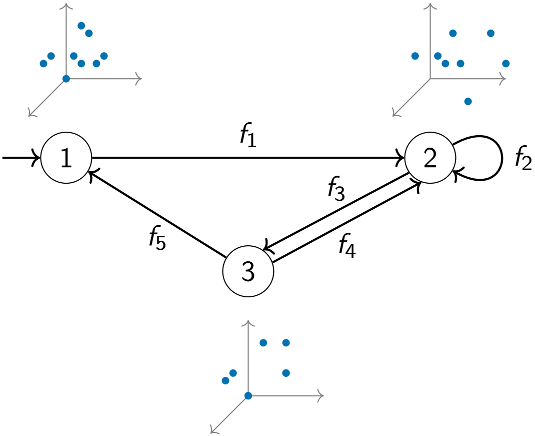

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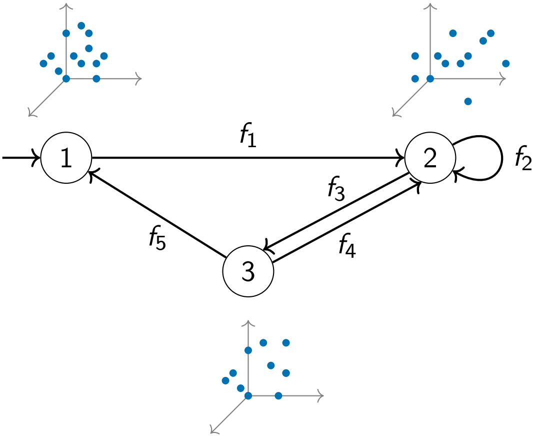

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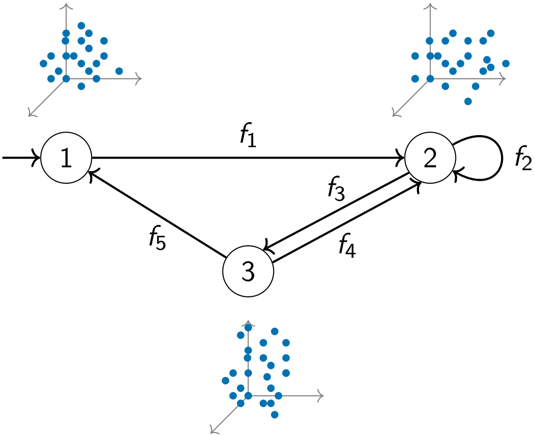

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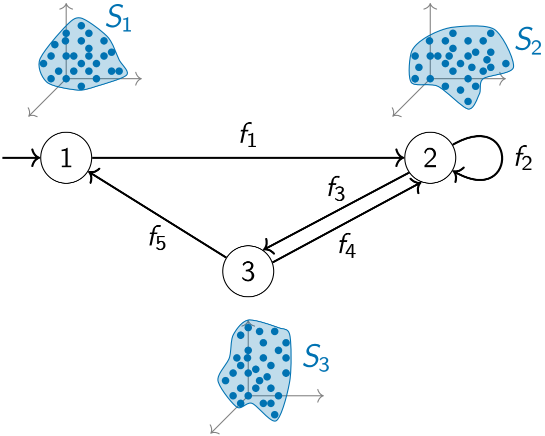

S_1,S_2,S_3 are the reachable states

Inductive invariants: example

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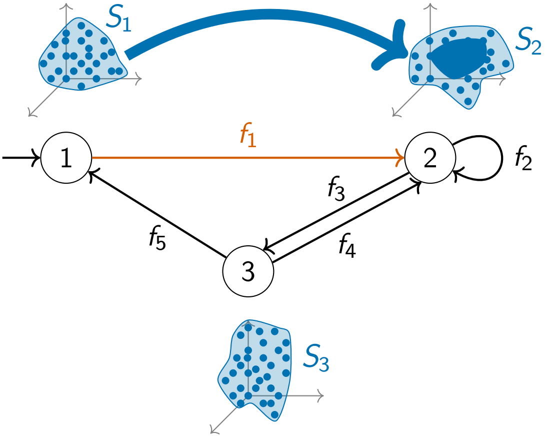

S_1, S_2, S_3 is also an inductive invariant

Inductive invariants: example

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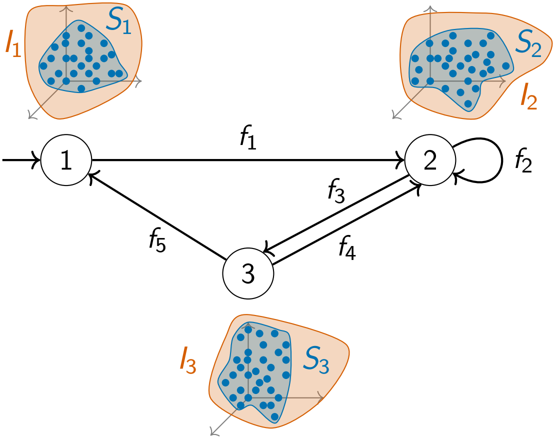

I_1, I_2, I_3 is an invariant

Inductive invariants: example

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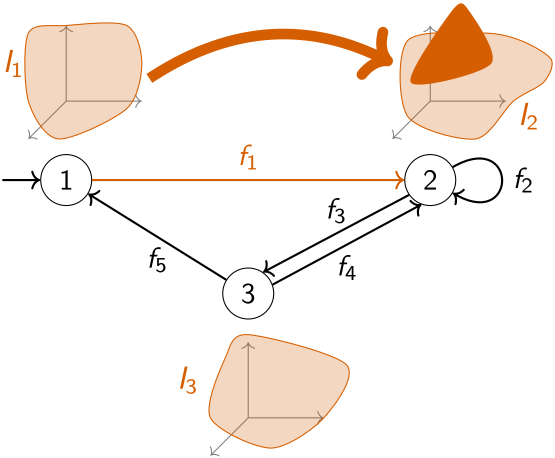

I_1, I_2, I_3 is **NOT** an inductive invariant

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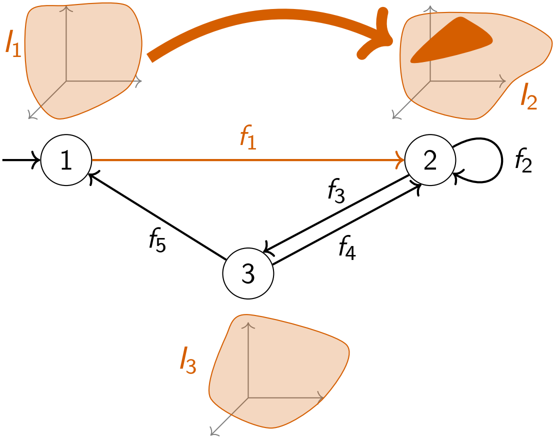
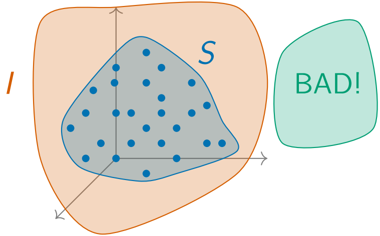


I_1, I_2, I_3 is an inductive invariant

Why Invariants?



The classical approach to the verification of temporal safety properties of programs requires the construction of ***inductive invariants*** *[...].* ***Automation of this construction is the main challenge in program verification.***

D. Beyer, T. Henzinger, R. Majumdar, and A. Rybalchenko
Invariant Synthesis for Combined Theories, 2007

Which invariants?