The Membership Problem for Hypergeometric Sequences with Rational Parameters

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A HYPERGEOMETRIC SEQUENCE is a sequence $a_1, a_2, \ldots, a_n$ of rational numbers satisfying a recurrence of the form

$$p_n u_{n+1} - p_{n-1} u_n = r_n u_n$$

POLYNOMIALLY RECURSIVE RATIONAL PARAMETERS $p_n, g_n, x_n$ have rational roots.

Catalan numbers:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$$

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$
Catalan numbers:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]

\[ C_0 = 1 \text{ and } C_{n+1} = \frac{2(2n+1)}{n+2} C_n \]
Hypergeometric sequences

A hypergeometric sequence is a sequence \( \langle u_0, u_1, u_2, \ldots \rangle \) of rational numbers satisfying a recurrence of the form

\[
p(n)u_n - q(n)u_{n-1} = 0
\]

Polynomially recursive
A hypergeometric sequence is a sequence \(<u_0, u_1, u_2, ...>\) of rational numbers satisfying a recurrence of the form:

\[ p(n)u_n - q(n)u_{n-1} = 0 \]

where:

- \(p(n)\) and \(q(n)\) are polynomials.

The sequence is polynomials recursively.

The shift quotient is given by:

\[ r(x) = \frac{p(x)}{q(x)} \]
\[ \langle \eta_{n} \rangle_{n=0}^{\infty} \quad \eta_0 = 1 \quad r(x) = \frac{(x + \frac{9}{2})(x + \frac{2}{2})(x + \frac{5}{2})}{(x + \frac{11}{2})(x + 4)(x + 1)} \]

1, 1.333, 1.588, 1.789, 1.951, 2.084, 2.195, ...
\[ \langle m_n \rangle_{n=0}^{\infty} \quad n_0 = 1 \quad r(x) = \frac{(x + \frac{9}{2})(x + \frac{3}{2})(x + \frac{5}{2})}{(x + 11)(x + 4)(x + 1)} \]

1, 1.333, 1.588, 1.789, 1.951, 2.084, 2.195, ...

Does the value \( \frac{13}{6} \approx 2.167 \) appear in the sequence?
\[
\langle u_n \rangle_{n=0}^\infty \quad u_0 = 1 \quad r(x) = \frac{(x + \frac{9}{2})(x + \frac{3}{2})(x + \frac{5}{2})}{(x + \frac{11}{2})(x + 4)(x + 1)}
\]

1, 1.333, 1.588, 1.789, 1.957, 2.084, 2.195, ...

Does the value \frac{13}{6} \approx 2.167 appear in the sequence?

**Prob. (MP)**

Given \( u_0 \in \mathbb{R} \), \( r(x) \in \mathbb{Q}(x) \) and \( t \in \mathbb{R} \):

\( \exists n \text{ s.t. } u_n = t \)?
Thm. (Skolem–Mahler–Lech)

The set of zeros of $\langle w_n \rangle_{n=0}^\infty$ is a union of finitely many arithmetic progressions and a finite set.
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Prob. (Skolem)
$\exists n \in \mathbb{N}$ s.t. $w_n = 0$?
Thm. (Skolem–Mahler–Lech)
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Prob. (Skolem)
\[ \exists n \in \mathbb{N} \text{ s.t. } w_n = 0 \quad ? \]

- order $\leq 4$: decidable
  - (Vereshchagin '85)
  - Mignotte, Shorey, Tijdeman '84
- order $= 5$: open!
Thm. (Skolem-Mahler-Lech) 
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* order = 4: decidable
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* order = 5: open!

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Thm. (Bell, Burris, Yeats '12)
The set of zeros of \( \langle u_n \rangle_{n=0}^{\infty} \) is a union of finitely many arithmetic progressions and a finite set.

* side condition: \( \rho(\kappa) = \text{const.} \)
**LRS** \( \langle w_n \rangle_{n=0}^{\infty} \)

**Thm. (Skolem–Mahler–Lech)**

The set of zeros of \( \langle w_n \rangle_{n=0}^{\infty} \) is a union of finitely many arithmetic progressions and a finite set.

**Prob. (Skolem)**

\[
\exists n \in \mathbb{N} \text{ s.t. } w_n = 0 ?
\]

\* order \( \leq 4 \): decidable

(Mignotte, Shorey, Tijdeman '84)

\* order \( 5 \): open!

---

**PRS** \( \langle u_n \rangle_{n=0}^{\infty} \)

**Thm. (Bell, Burris, Yeats '12)**

The set of zeros of \( \langle u_n \rangle_{n=0}^{\infty} \) is a union of finitely many arithmetic progressions and a finite set.

**Prob.**

\[
\exists n \in \mathbb{N} \text{ s.t. } u_n = 0 ?
\]

\* side condition: \( \forall x \in \mathbb{R} \), \( \exists \text{ const.} \)
**LRS** $\langle w_n \rangle_{n=0}^\infty$

**Thm. (Skolem-Mahler-Lech)**
The set of zeros of $\langle w_n \rangle_{n=0}^\infty$ is a union of finitely many arithmetic progressions and a finite set.

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The set of zeros of $\langle u_n \rangle_{n=0}^\infty$ is a union of finitely many arithmetic progressions and a finite set.

**Prob.**
$\exists n \in \mathbb{N} \text{ s.t. } u_n = 0$?

*order $1$: trivial

*order $2$: open!

*side condition: $P(u(x)) = \text{const.}$
Thm. (Skolem-Mahler-Lech)
The set of zeros of \( \langle w_n \rangle_{n=0}^\infty \) is a union of finitely many arithmetic progressions and a finite set.

**Prob. (Skolem)**
\[
\exists n \in \mathbb{N} \text{ s.t. } w_n = 0 ?
\]

*order \( \leq 4 \): decidable
(Mignotte, Shorey, Tijdeman '84)

*order 5: open!

**PRS** \( \langle u_n \rangle_{n=0}^\infty \)

Thm. (Bell, Burris, Yeats '12)
The set of zeros of \( \langle u_n \rangle_{n=0}^\infty \) is a union of finitely many arithmetic progressions and a finite set.

**Prob.**
\[
\exists n \in \mathbb{N} \text{ s.t. } u_n = 0 ?
\]

*order 1: trivial
*sum of two HS \( \leftrightarrow \) MP

*order 2: open!

*side condition: \( P(u) = \text{const.} \)
Decidability

Thm.

The Membership Problem for hypergeometric sequences with rational parameters is decidable.
Decidability

Thm.

The Membership Problem for hypergeometric sequences with rational parameters is decidable.

Find \( M \in \mathbb{N} \) s.t. for all \( n > M \):

\[ u_n \neq t. \]
Decidability

Thm.

The Membership Problem for hypergeometric sequences with rational parameters is decidable.

Find \( \text{MEIN} \) s.t. for all \( n > M \):

\[ u_n \neq t. \]

How?

Divisibility by primes!

MP \[\rightarrow\] finite search in \( u_0, \ldots, u_M \)
The problem is (almost) trivial ...

\[ r(x) \text{ as } x \to \infty \text{ converges to } e \in \mathbb{Q} \cup \{\pm \infty\} \]
The problem is (almost) trivial ...

\( r(x) \) as \( x \to \infty \) converges to \( \lambda \in \mathbb{Q} \cup \{ \pm \infty \} \)

\( l \neq \pm 1: \)

\[ \exists M \in \mathbb{N} \; \text{s.t.} \]

\[ |u_n| = |u_0| \prod_{k=1}^{n} r(k) > |t| \quad \text{for all } n \geq M \]
The problem is (almost) trivial ...

\[ r(x) \text{ as } x \to \infty \text{ converges to } \ell \in \mathbb{Q} \cup \{\pm \infty\} \]

- \( \ell \neq \pm 1 \):

  \[ \exists \, M \in \mathbb{N} \text{ s.t.} \]

  \[ |u_n| = |u_0| \prod_{k=1}^{n} r(k) > |t| \text{ for all } n \geq M \]

- \( \ell = \pm 1 \):

  divisibility by primes
\[
\langle n_n \rangle_{n=0}^\infty \quad n_0 = 1 \\
\]

\[
r(x) = \frac{(x + \frac{9}{2})(x + \frac{3}{2})(x + \frac{5}{2})}{(x + \frac{11}{2})(x + 4)(x + 1)}
\]

\[
\exists n \in \mathbb{N} \text{ s.t. } n = \frac{13}{6} \quad ? \]

\[
\frac{13}{6} \approx 2.167
\]

1, 1.333, 1.588, 1.789, 1.951, 2.084, 2.195, ...
\[ \langle u_n \rangle_{n=0}^\infty \quad u_0 = 1 \quad r(x) = \frac{(x+\frac{9}{2})(x+\frac{9}{2})(x+\frac{5}{2})}{(x+\frac{11}{2})(x+4)(x+1)} \]

\[ \exists n \in \mathbb{N} \text{ s.t. } u = \frac{13}{6} \quad \frac{13}{6} \approx 2.167 \]

1, 1.333, 1.588, 1.789, 1.951, 2.084, 2.195, ...

\[ \lim_{n \to \infty} u_n = \frac{3 \cdot 2^5}{5^{11}} \neq \frac{13}{6} \]
A number theoretical diversion

\[ u_n = u_0 \frac{n}{1} \prod_{k=1}^{n} r(k) = u_0 \frac{n}{1} \prod_{k=1}^{n} \frac{(k+\alpha_1) \cdots (k+\alpha_d)}{(k+\beta_1) \cdots (k+\beta_d)} \]

\[ n \to \infty ? \]
A number theoretical diversion

\[ u_n = u_0 \frac{n}{\prod_{k=1}^{n}} \quad r(k) = u_0 \frac{n}{\prod_{k=1}^{n}} \frac{(k+\alpha_1) \cdots (k+\alpha_d)}{(k+\beta_1) \cdots (k+\beta_d)} \]

\[ n \to \infty ? \]

\[ \Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{n}{n+z} \left(1 + \frac{1}{z}\right)^z \]

\[ \Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{N} \]
A number theoretical diversion

\[ u_n = u_0 \prod_{k=1}^{n} \frac{k}{(k+\alpha_1) \cdots (k+\alpha_d)} \]

\[ r(k) = \frac{n}{n+\beta_1} \prod_{k=1}^{\infty} \frac{(k+\alpha_1) \cdots (k+\alpha_d)}{(k+\beta_1) \cdots (k+\beta_d)} \]

\[ n \to \infty ? \]

\[ \Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{n}{n+z} \left(1 + \frac{1}{n} \right)^z \quad \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N} \]

Prop.

Let \( \alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}. \)

If \( \alpha_1 + \cdots + \alpha_d = \beta_1 + \cdots + \beta_d \) then

\[ \prod_{k=1}^{\infty} \frac{(k+\alpha_1) \cdots (k+\alpha_d)}{(k+\beta_1) \cdots (k+\beta_d)} = \frac{\beta_1 \cdots \beta_d}{\alpha_1 \cdots \alpha_d} \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)}{\Gamma(\beta_1) \cdots \Gamma(\beta_d)} \]

otherwise the infinite product diverges.
A number theoretical diversion

Prop.

The Membership Problem for hypergeometric sequences reduces to deciding, given $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$, whether

$$\prod_{k=1}^{d} \Gamma(\beta_k) = \prod_{k=1}^{d} \Gamma(\alpha_k)$$
A number theoretical diversion

Prop.

The Membership Problem for hypergeometric sequences reduces to deciding, given \( \alpha_1, \ldots, \alpha_d \) and \( \beta_1, \ldots, \beta_d \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \), whether

\[
\Gamma(\beta_1) \cdots \Gamma(\beta_d) = \Gamma(\alpha_1) \cdots \Gamma(\alpha_d)
\]

Values of the Gamma function?
A number theoretical diversion

Prop.
The Membership Problem for hypergeometric sequences reduces to deciding, given $\alpha_1, \ldots, \alpha_d$ and $(\beta_1, \ldots, \beta_d) \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$, whether

$$\prod (\beta_1) \ldots \prod (\beta_d) = \prod (\alpha_1) \ldots \prod (\alpha_d)$$

Values of the Gamma function?

$$\frac{\Gamma \left( \frac{1}{14} \right) \Gamma \left( \frac{3}{14} \right) \Gamma \left( \frac{11}{14} \right)}{\Gamma \left( -\frac{3}{14} \right) \Gamma \left( \frac{5}{14} \right) \Gamma \left( \frac{12}{14} \right)} = 2$$
A number theoretical diversion

Prop.
The Membership Problem for hypergeometric sequences reduces to deciding, given \(a_1, \ldots, a_d\) and \((\beta_1, \ldots, \beta_d) \in \mathbb{R}\setminus\mathbb{Z}_{\geq 0}\), whether
\[
\prod (\beta_1) \cdots \prod (\beta_d) = \prod (a_1) \cdots \prod (a_d)
\]

Values of the Gamma function:
\[
\frac{\Gamma \left( \frac{1}{14} \right) \Gamma \left( \frac{3}{14} \right) \Gamma \left( \frac{9}{14} \right)}{\Gamma \left( \frac{3}{14} \right) \Gamma \left( \frac{5}{14} \right) \Gamma \left( \frac{11}{14} \right)} = 2
\]

Conj. (Rohrlich-Lang '78)
Any multiplicative relation of Gamma values on rational points that is algebraic is a consequence of the standard relations translation, reflection, multiplication.
Using divisibility

Find $M \in \mathbb{N}$ s.t. for all $n > M$, $a_n \neq t$. 
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Find $M \in \mathbb{N}$ s.t. for all $n > M$, $a_n \neq t$.

$p$-adic valuation

$\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{ \infty \}$

$\nu_p(x) = k$ if $x = p^k \frac{a}{b}$ and $p \nmid b$

$\nu_p(0) = \infty$
Using divisibility

Find \( M \in \mathbb{N} \) s.t. for all \( n > M \), \( a_n \neq t \).

\[ p \text{-adic valuation} \]

\[ n_p : \mathbb{Q} \to \mathbb{Z} \cup \{ \infty \} \]

\[ n_3 \left( \frac{36}{5} \right) = n_3 \left( 9 \cdot \frac{4}{5} \right) = 2 \]

\[ n_p (x) := k \text{ if } x = p^k \cdot \frac{a}{b} \text{ and } p \nmid b \]

\[ n_p (0) := \infty \]

\[ n_p (a \cdot b) = n_p (a) + n_p (b) \]
Using divisibility

\[ \text{Find } M \in \mathbb{N} \text{ s.t. for all } n > M, \ a_n \neq t. \]

\[ p \text{-adic valuation} \]

\[ \nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{ \infty \} \]

\[ \nu_3 \left( \frac{36}{5} \right) = \nu_3 (9 \cdot \frac{4}{5}) = 2 \]

\[ \nu_p (x) := k \quad \text{if } x = p^k \frac{a}{b} \quad \text{and } \ p \nmid b \]

\[ \nu_p (0) := \infty \]

\[ \nu_p (a \cdot b) = \nu_p (a) + \nu_p (b) \]

\[ \exists M \in \mathbb{N} \text{ s.t. for all } n > M \quad \exists \text{prime } p \text{ s.t.} \]

\[ \nu_p (a_n) = 0 \quad \text{and} \quad \nu_p (t) = 0 \]
\[ \langle \nu_n \rangle_{n=0}^{\infty} \]

\[ \nu_0 = 1 \]

\[ r(x) = \frac{(x + \frac{9}{2})(x + \frac{3}{2})(x + \frac{5}{2})}{(x + \frac{11}{2})(x + 4)(x + 1)} \]

\[ \nu_n = 1 \cdot \prod_{k=1}^{n} \frac{(k + \frac{9}{2})(k + \frac{3}{2})(k + \frac{5}{2})}{(k + \frac{11}{2})(k + 4)(k + 1)} \]
\[ \langle n_n \rangle_{n=0}^\infty \]

\[ w_0 = 1 \]

\[ r(x) = \frac{(x + \frac{9}{2})(x + \frac{3}{2})(x + \frac{5}{2})}{(x + \frac{11}{2})(x + 4)(x + 1)} \]

\[ w_n = 1 \cdot \prod_{k=1}^{n} \frac{(k + \frac{9}{2})(k + \frac{3}{2})(k + \frac{5}{2})}{(k + \frac{11}{2})(k + 4)(k + 1)} \]

\[ \nu_p (n_n) = \sum_{k=1}^{n} \left[ \left( \nu_p (k+\frac{9}{2}) + \nu_p (k+\frac{3}{2}) + \psi_p (\frac{5}{2}) \right) - \left( \nu_p (k+\frac{11}{2}) + \nu_p (k+4) + \nu_p (k+1) \right) \right] \]
\[ \langle u_n \rangle_{n=0}^{\infty} \quad u_0 = 1 \quad r(x) = \frac{(x + \frac{9}{2})(x + \frac{3}{2})(x + \frac{5}{2})}{(x + \frac{11}{2})(x + 4)(x + 1)} \]

\[ u_n = 1 \cdot \prod_{k=1}^{n} \frac{(k + \frac{9}{2})(k + \frac{3}{2})(k + \frac{5}{2})}{(k + \frac{11}{2})(k + 4)(k + 1)} \]

\[ \nu_p(u_n) = \sum_{k=1}^{n} \left[ \left( \nu_p(k + \frac{9}{2}) + \nu_p(k + \frac{3}{2}) + \nu_p(\frac{5}{2}) \right) - \left( \nu_p(k + \frac{11}{2}) + \nu_p(k + 4) + \nu_p(k + 1) \right) \right] \]

\[ \text{prime } 17 \]

\[ \sum_{\text{roots of } r} \nu_p \neq \sum_{\text{poles of } r} \nu_p \]
prime $17 \quad \Sigma_{a_P} \neq \Sigma_{a_P}$
forall n > 5, \( u_n \neq t \) → check only \( \{u_0, u_1, u_2, u_3, u_4\} \)
The result

Thm.

The Membership Problem for hypergeometric sequences with rational parameters is decidable.

Proof:

$p$-adic valuations + results on the density of primes
The result

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Proof:

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Ongoing work:

general algebraic parameters