The Membership Problem for Hypergeometric Sequences with Rational Parameters

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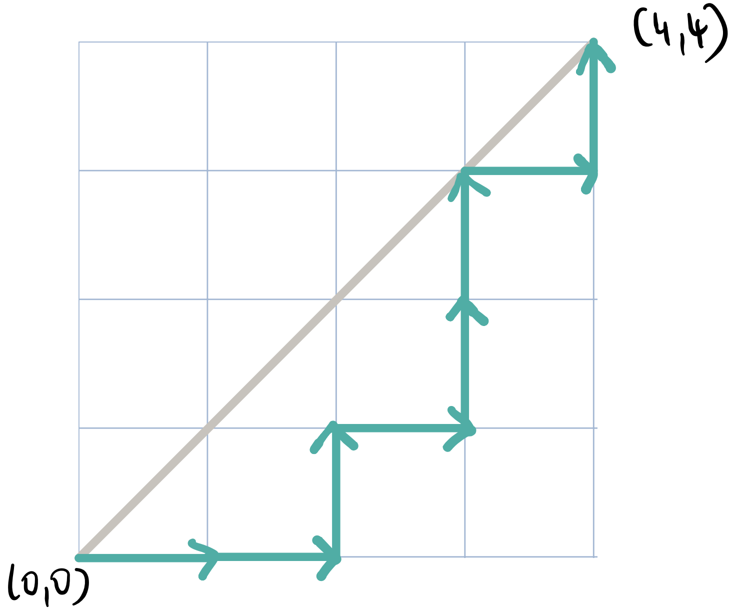
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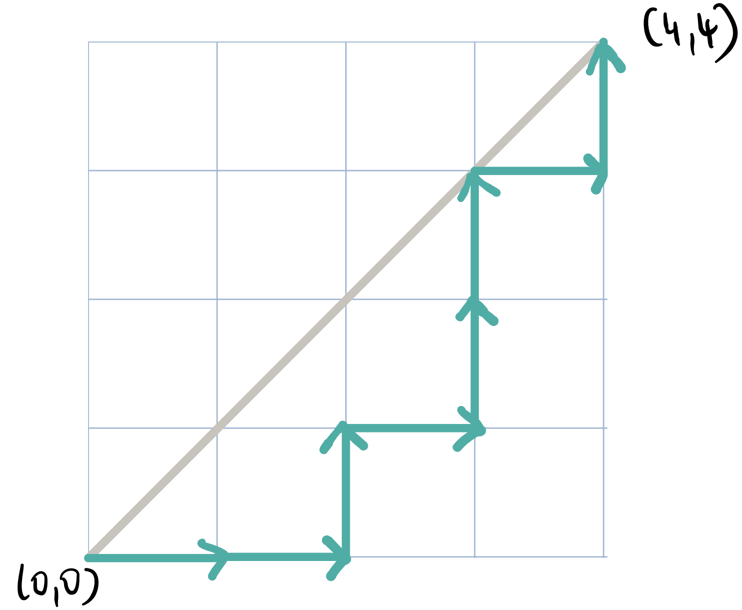

Catalan numbers:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...

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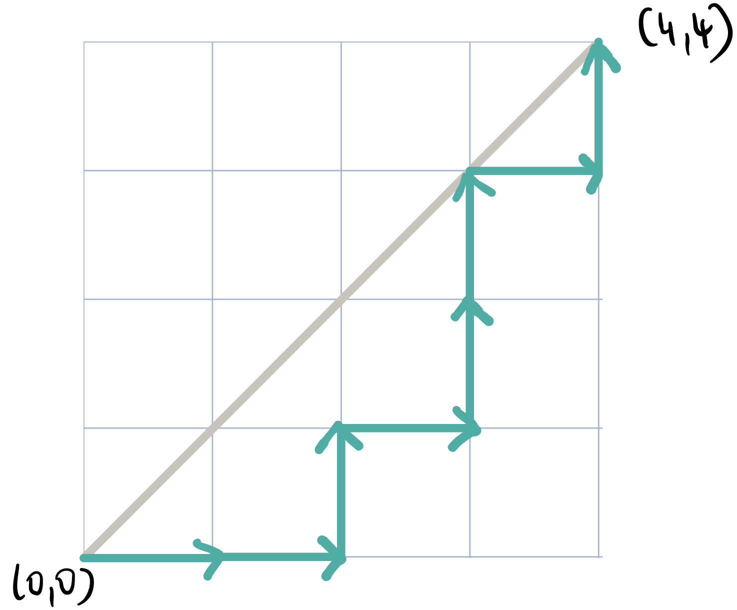

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$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n$$

Hypergeometric sequences

* A HYPERGEOMETRIC SEQUENCE is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of rational numbers satisfying a recurrence of the form

$$\boxed{\underbrace{p(n)u_n - q(n)u_{n-1} = 0}_{\text{POLYNOMIALLY RECURSIVE}}}$$

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$$\boxed{u_n = r(n)\, u_{n-1}}$$

shift quotient $\quad r(x) = \dfrac{p(x)}{Q(x)}$

$$\langle u_n \rangle_{n=0}^{\infty} \qquad u_0 = 1 \qquad r(x) = \frac{\left(x+\frac{9}{2}\right)\left(x+\frac{7}{2}\right)\left(x+\frac{5}{2}\right)}{\left(x+\frac{11}{2}\right)(x+4)(x+1)}$$

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Prob. (MP)

Given $u_0 \in \mathbb{Q}$, $r(x) \in \mathbb{Q}(x)$ and $t \in \mathbb{Q}$:

$$\exists n \text{ s.t. } u_n = t \ ?$$

LRS $\langle w_n \rangle_{n=0}^{\infty}$

Thm. (Skolem-Mahler-Lech)
The set of zeros of $\langle w_n \rangle_{n=0}^{\infty}$ is a union of finitely many arithmetic progressions and a finite set.

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- order 5: open!

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* side condition: $p_d(x) = \text{const}$.

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- sum of two HS $\longleftrightarrow$ MP
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How?

Divisibility by primes!

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$$\lim_{n \to \infty} u_n = \frac{3 \cdot 2^5}{5\pi} \neq \frac{13}{6}$$

A number theoretical diversion

$$u_n = u_0 \prod_{k=1}^{n} r(k) = u_0 \prod_{k=1}^{n} \frac{(k+\alpha_1)\cdots(k+\alpha_d)}{(k+\beta_1)\cdots(k+\beta_d)}$$

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$$n \to \infty ?$$

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{n}{n+z}\left(1+\frac{1}{z}\right)^z \qquad \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N}$$

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Prop. Let $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$.

If $\alpha_1 + \dots + \alpha_d = \beta_1 + \dots + \beta_d$ then

$$\prod_{k=1}^{\infty} \frac{(k+\alpha_1)\cdots(k+\alpha_d)}{(k+\beta_1)\cdots(k+\beta_d)} = \frac{\beta_1 \cdots \beta_d}{\alpha_1 \cdots \alpha_d} \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_d)}{\Gamma(\beta_1)\cdots\Gamma(\beta_d)}$$

otherwise the infinite product diverges.

g

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Prop. The Membership Problem for hypergeometric sequences reduces to deciding, given $\alpha_1, \dots, \alpha_d$ and $\beta_1, \dots, \beta_d \in \mathbb{R} \setminus \mathbb{Z}_{<0}$, whether

$$\Gamma(\beta_1) \cdots \Gamma(\beta_d) = \Gamma(\alpha_1) \cdots \Gamma(\alpha_d)$$

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$$\frac{\Gamma(\frac{1}{14})\,\Gamma(\frac{9}{14})\,\Gamma(\frac{11}{14})}{\Gamma(\frac{3}{14})\,\Gamma(\frac{5}{14})\,\Gamma(\frac{13}{14})} = 2$$

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Conj. (Rohrlich–Lang '78) Any multiplicative relation of Gamma values on rational points that is algebraic is a consequence of the standard relations

translation, reflection, multiplication.

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Find $M \in \mathbb{N}$ s.t. for all $n > M$, $u_n \neq t$.

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$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$

$v_p(x) := k$ if $x = p^k \frac{a}{b}$ and $p \nmid ab$

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$\exists M \in \mathbb{N}$ s.t. for all $n > M$ $\exists$ prime p s.t.

$v_p(u_n) \neq 0$ and $v_p(t) = 0$

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$$\nu_p(u_n) = \sum_{k=1}^{n} \left[\left(\nu_p\left(k+\frac{9}{2}\right) + \nu_p\left(k+\frac{7}{2}\right) + \nu_p\left(\frac{5}{2}\right) \right) - \left(\nu_p\left(k+\frac{11}{2}\right) + \nu_p(k+4) + \nu_p(k+1) \right) \right]$$

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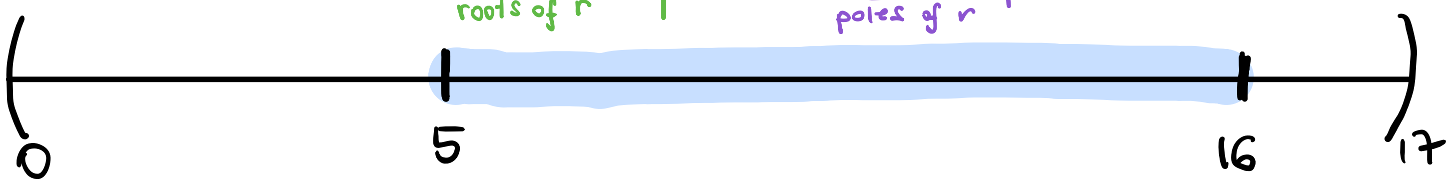
prime 17

$$\sum_{\text{roots of } r} \nu_p \neq \sum_{\text{poles of } r} \nu_p$$



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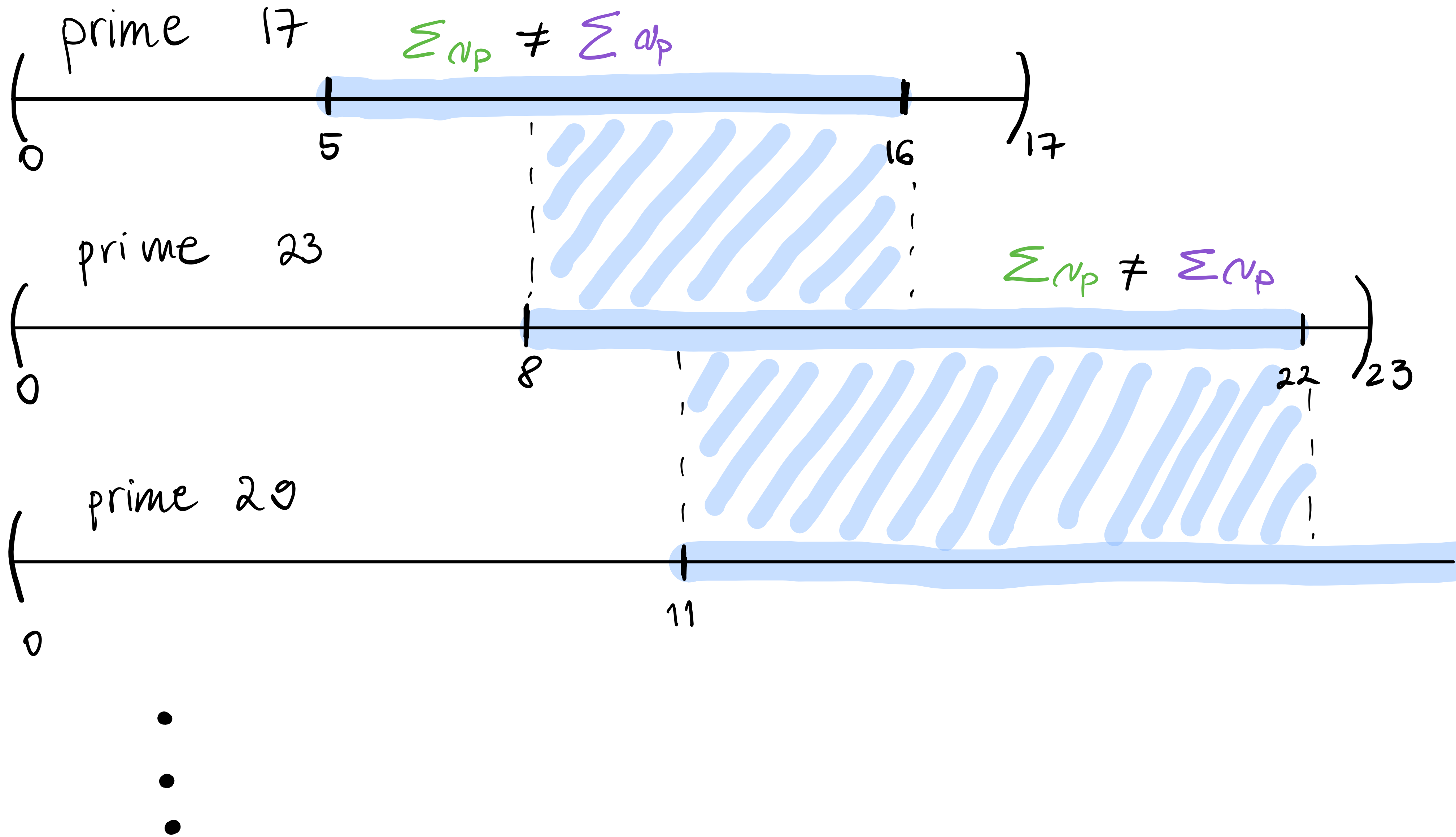
$\sum n_p \neq \sum a_p$



$\forall n > 5, \; u_n \neq t \longrightarrow$ check only $\{u_0, u_1, u_2, u_3, u_4\}$

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Proof:

p-adic valuations + results on the density of primes

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Ongoing work:

general algebraic parameters