

A (truly) universal differential equation

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Joint work with Olivier Bournez and Daniel Graça

21 june 2019



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Cité des Sciences



What is a computer ?

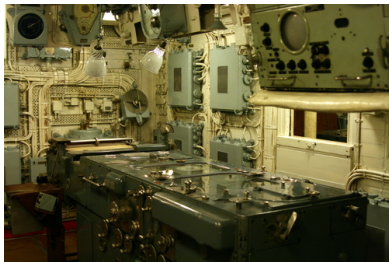
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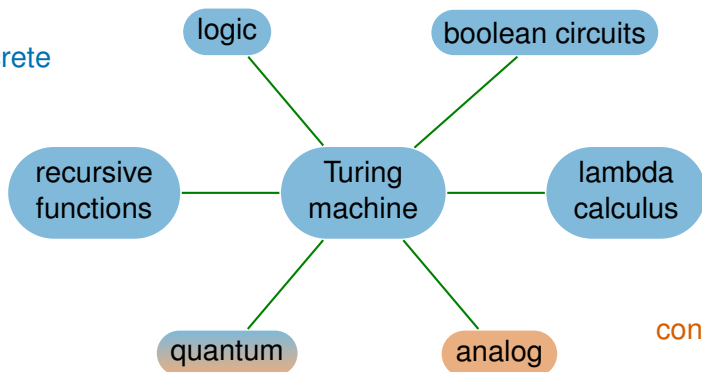


VS



Computability

discrete



continuous

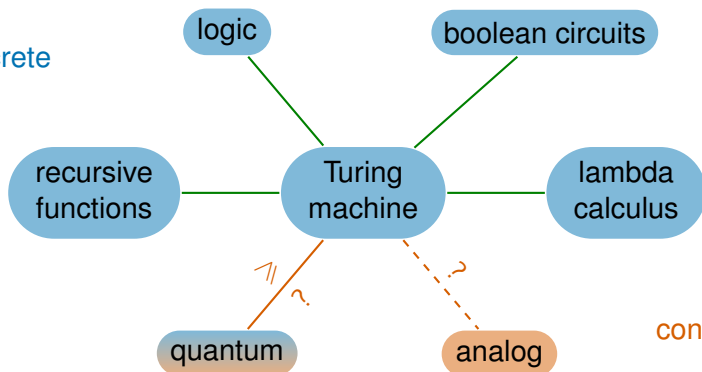
Church Thesis

All **reasonable** models of computation are equivalent.

Church Thesis

Complexity

discrete



continuous

Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.

Polynomial Differential Equations



Differential Analyzer

Polynomial Differential Equations

$$\boxed{k} \rightarrow k \quad \begin{matrix} u \\ v \end{matrix} \rightarrow \boxed{\times} \rightarrow uv$$

$$\begin{matrix} u \\ v \end{matrix} \rightarrow \boxed{+} \rightarrow u+v \quad u \rightarrow \boxed{\int} \rightarrow \int u$$

General Purpose Analog
Computer, Shannon 1936



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Differential Analyzer

Polynomial differential
equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- ▶ Rich class
- ▶ Stable (+, ×, ∘, /, ED)
- ▶ No closed-form solution

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General Purpose Analog
Computer, Shannon 1936



Differential Analyzer

Newton mechanics

Reaction networks :

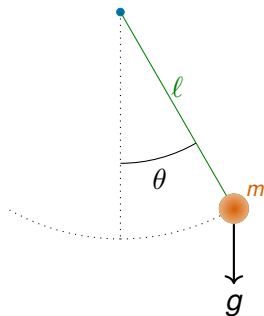
- ▶ chemical
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Polynomial differential
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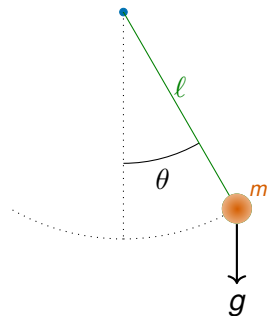
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Example of dynamical system



$$\ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0$$

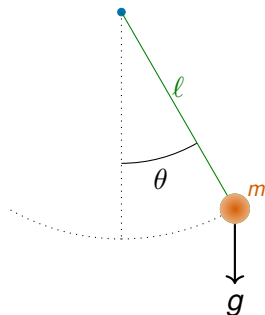
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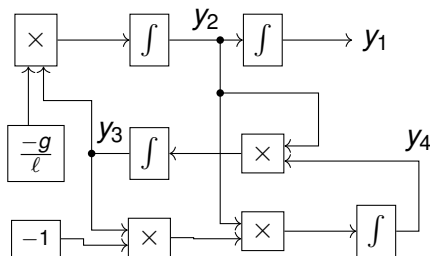
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$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{\ell} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

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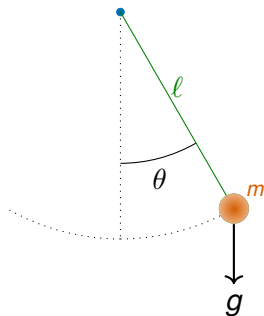


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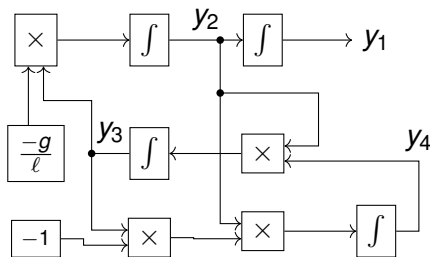


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Historical remark : the word “analog”

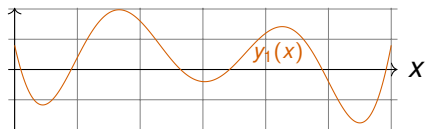
The pendulum and the circuit have the same equation. One can study one using the other by **analogy**.

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \quad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



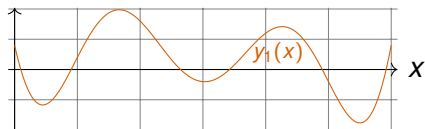
Shannon's notion

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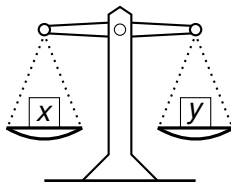
Shannon's notion

$\sin, \cos, \exp, \log, \dots$

Considered "weak" : not Γ and ζ
Only analytic functions

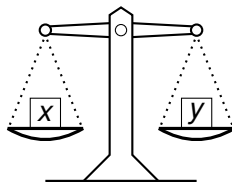
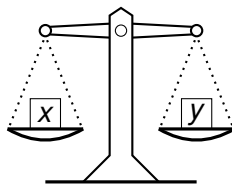
Does a balance scale compute a function?

Inputs : $x, y \in [0, +\infty)$



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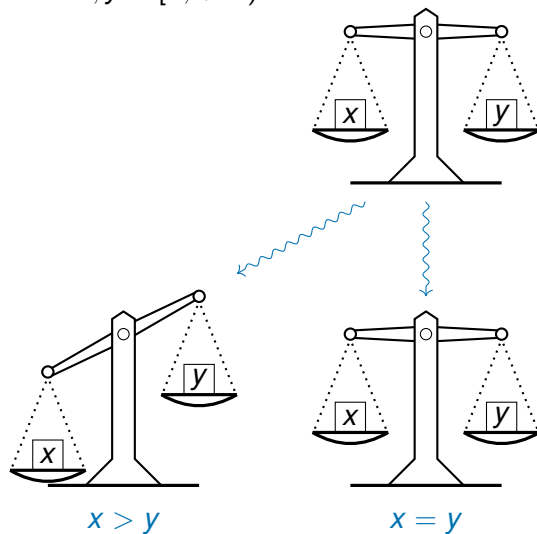
Inputs : $x, y \in [0, +\infty)$



$$x = y$$

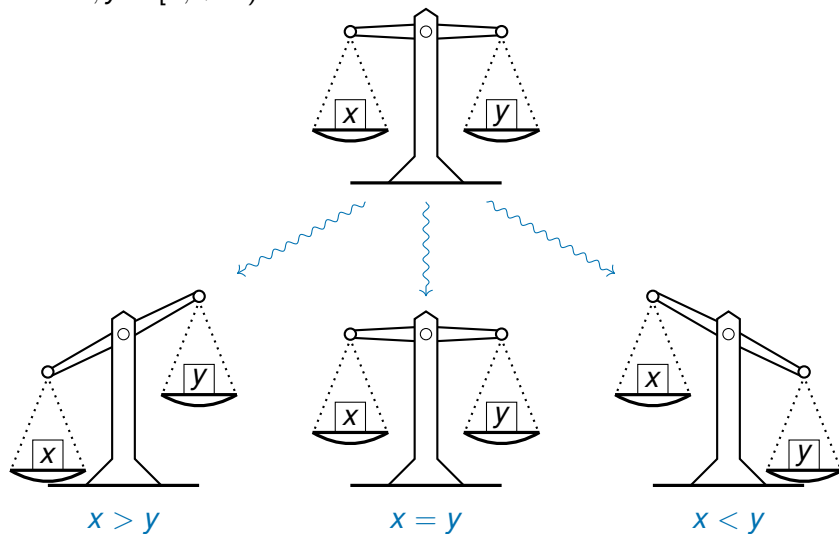
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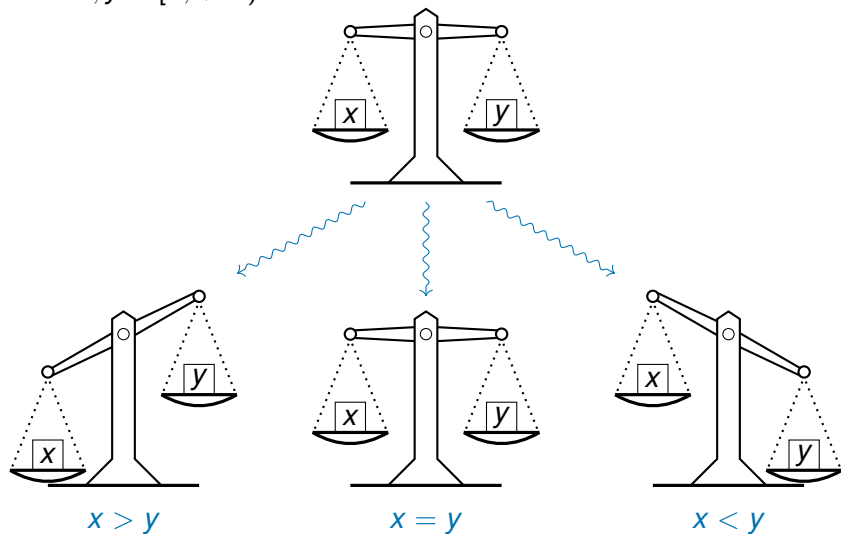
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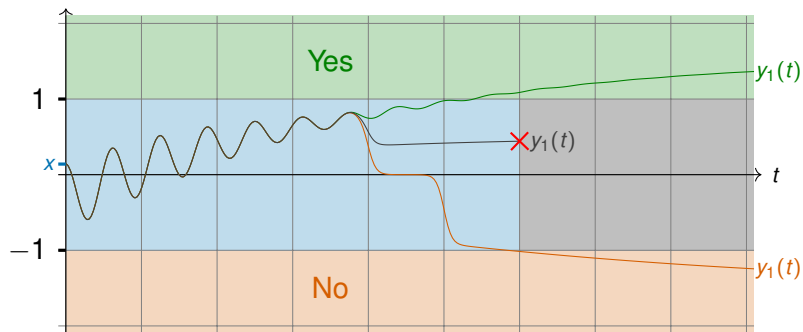
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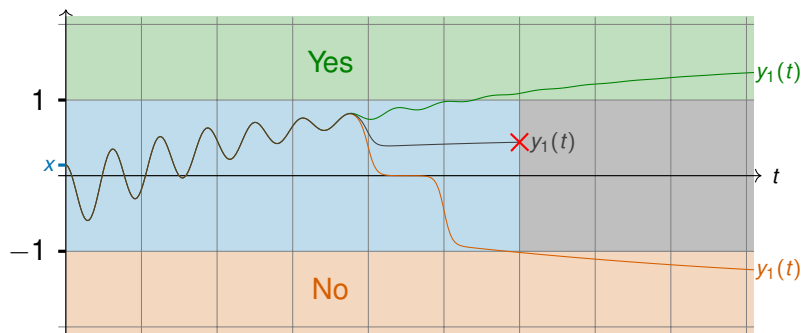


Output : $\text{sign}(x - y)$?

More formally



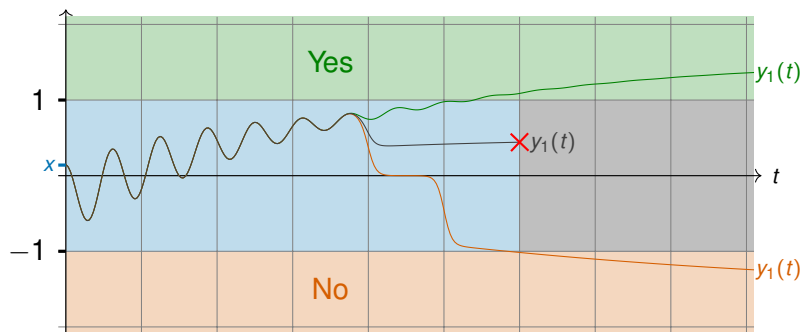
More formally



Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

More formally



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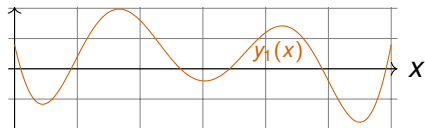
- ▶ analog computability theory
- ▶ purely continuous characterization of classical computability

Computing with differential equations (cont.)

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \quad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



Shannon's notion

sin, cos, exp, log, ...

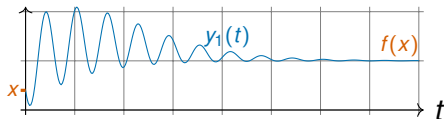
Considered "weak" : not Γ and ζ

Only analytic functions

Computable

$$\begin{cases} y(0) = q(x) \\ y'(t) = p(y(t)) \end{cases} \quad \begin{matrix} x \in \mathbb{R} \\ t \in \mathbb{R}_+ \end{matrix}$$

$$f(x) = \lim_{t \rightarrow \infty} y_1(t)$$



Modern notion

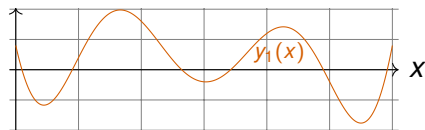
sin, cos, exp, log, Γ , ζ , ...

Turing powerful

[Bournez et al., 2007]

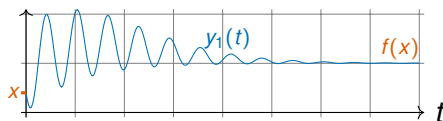
Universal differential equations

Generable functions



subclass of analytic functions

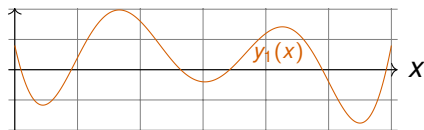
Computable functions



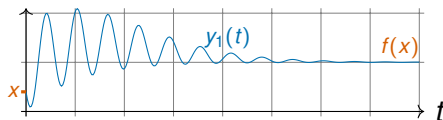
any computable function

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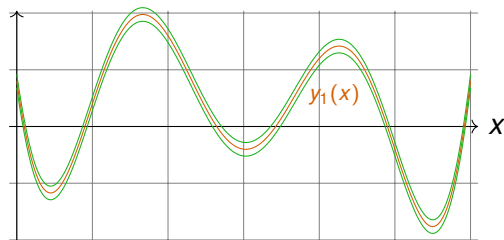


Computable functions

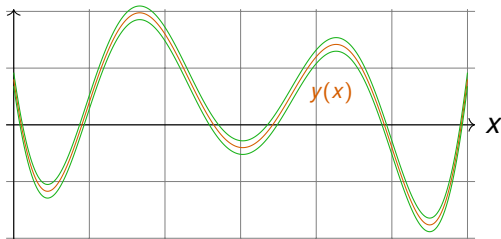


subclass of analytic functions

any computable function



Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

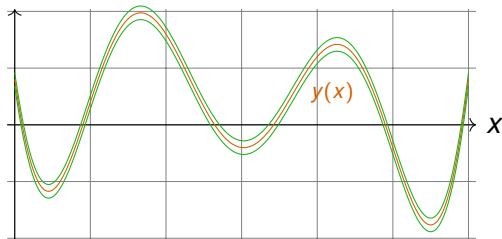
For any continuous functions f and ε , there exists $y : \mathbb{R} \rightarrow \mathbb{R}$ solution to

$$\begin{aligned} 3y'^4 y'' y''''^2 &- 4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ &- 12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leq \varepsilon(t).$$

Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

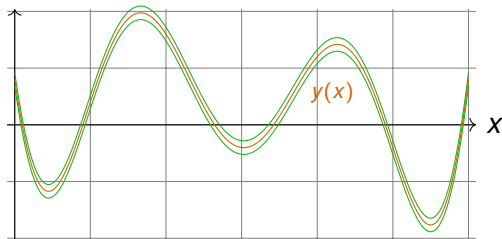
There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε , there exists a solution $y : \mathbb{R} \rightarrow \mathbb{R}$ to

$$p(y, y', \dots, y^{(4)}) = 0$$

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Problem : this is «weak» result.

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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- ▶ Rubel's statement : this DAE is universal
- ▶ More realistic interpretation : this DAE allows almost anything

Open Problem (Rubel, 1981)

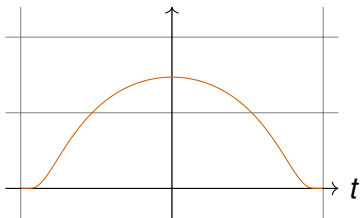
Is there a universal ODE $y' = p(y)$?

Note : explicit polynomial ODE \Rightarrow unique solution

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

It satisfies $(1 - t^2)^2 f''(t) + 2tf'(t) = 0$.



Rubel's proof in one slide

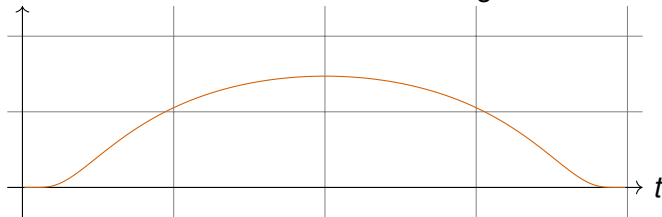
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- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at + b)$ satisfies

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Translation and rescaling :



Rubel's proof in one slide

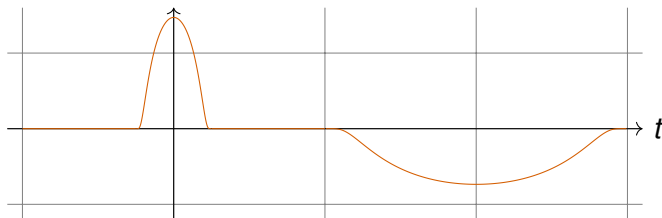
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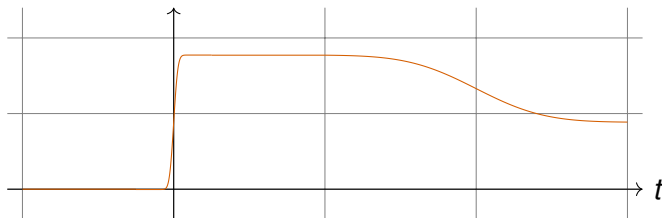
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- Can glue together arbitrary many such pieces
- Can arrange so that $\int f$ is solution : **piecewise pseudo-linear**



Rubel's proof in one slide

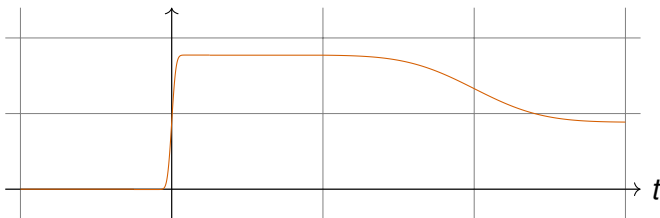
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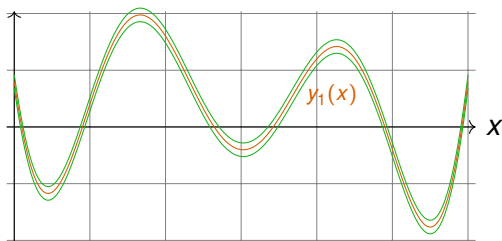
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Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in C^0**

Universal initial value problem (IVP)



Theorem

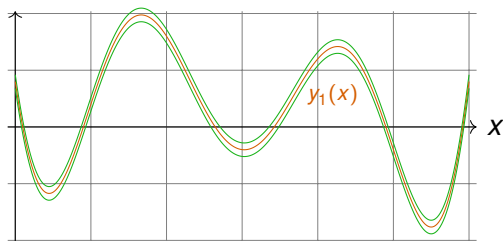
There exists a **fixed** (vector of) polynomial p such that for any continuous functions f and ε , there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \rightarrow \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$

Universal initial value problem (IVP)



Notes :

- ▶ **system** of ODEs,
- ▶ y is analytic,
- ▶ we need $d \approx 300$.

Theorem

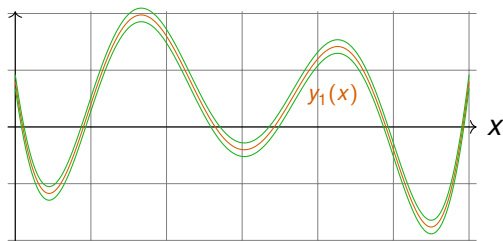
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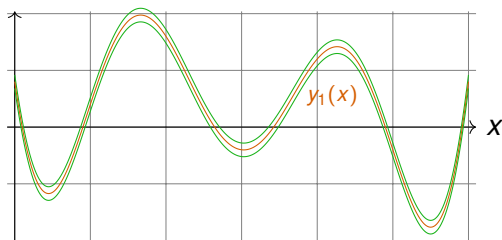
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Remark : α is usually transcendental, but computable from f and ε

Universal DAE revisited



Theorem

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$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** and this solution satisfies such that

$$|y(t) - f(t)| \leq \varepsilon(t).$$

A brief stop

Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by [programming with ODEs](#).

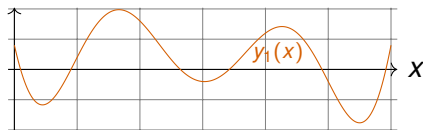
Generable functions : a summary

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if $\exists d, p$ and y_0 such that the solution y to

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satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.



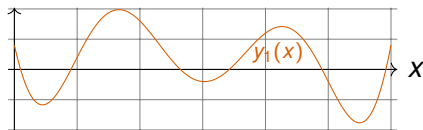
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Nice theory for the class of total and univariate **generable** functions :

- ▶ analytic
- ▶ contains polynomials, sin, cos, tanh, exp
- ▶ stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)

$$y' = f(y)$$

- ▶ solutions to polynomial ODEs form a **very large class**

Why is this useful ?

Writing polynomial ODEs by hand is **hard**.

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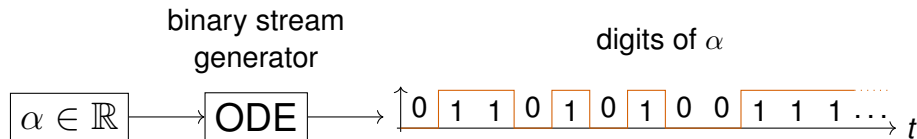
Using **generable functions**, we can build complicated **multivariate partial functions** using other operations, and we know they are solutions to polynomial ODEs **by construction**.

Example : almost rounding function

There exists a generable function round such that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $\lambda > 2$ and $\mu \geq 0$:

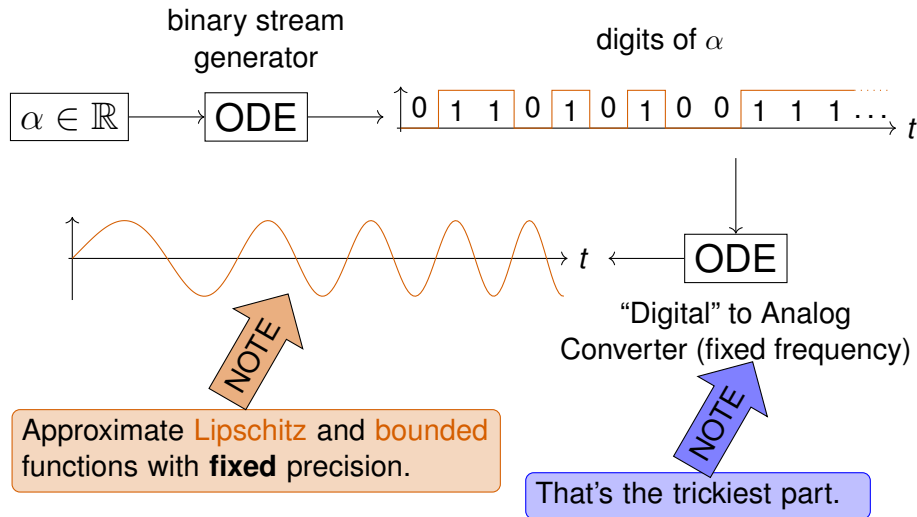
- ▶ if $x \in [n - \frac{1}{2}, n + \frac{1}{2}]$ then $|\text{round}(x, \mu, \lambda) - n| \leq \frac{1}{2}$,
- ▶ if $x \in [n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}]$ then $|\text{round}(x, \mu, \lambda) - n| \leq e^{-\mu}$.

A simplified proof

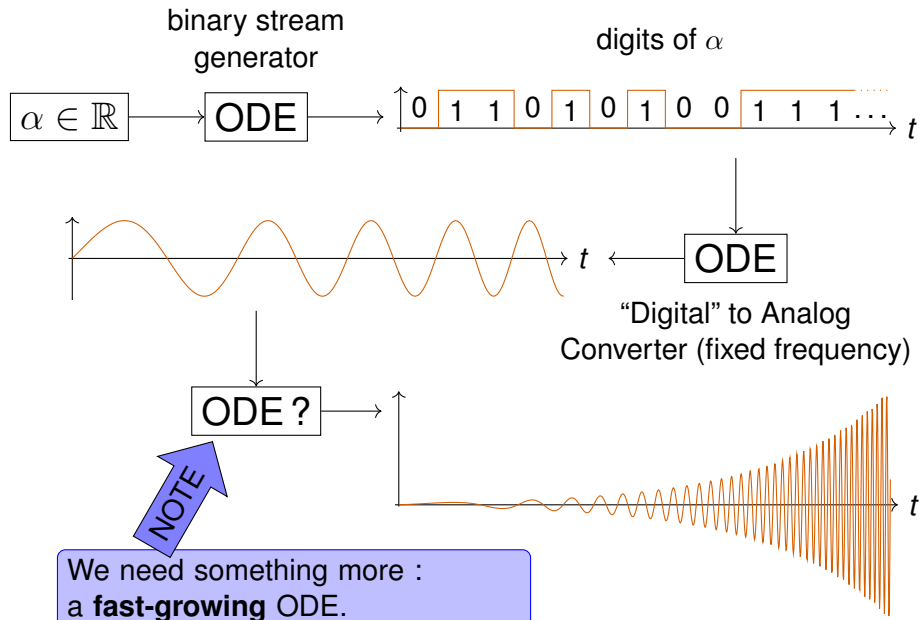


This is the **ideal** curve, the real one is an approximation of it.

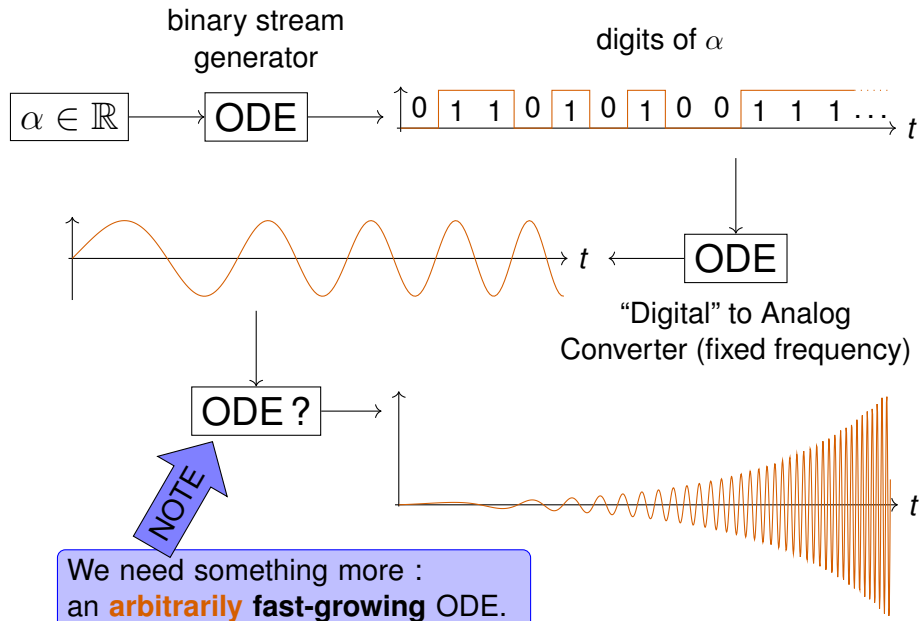
A simplified proof



A simplified proof



A simplified proof



A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



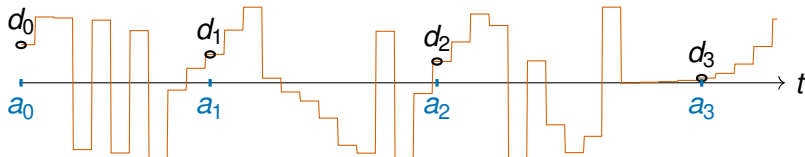
$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2} \tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4, \lambda)} + 4\pi/3))$$

It's horrible, but generable

round is the mysterious rounding function...

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$

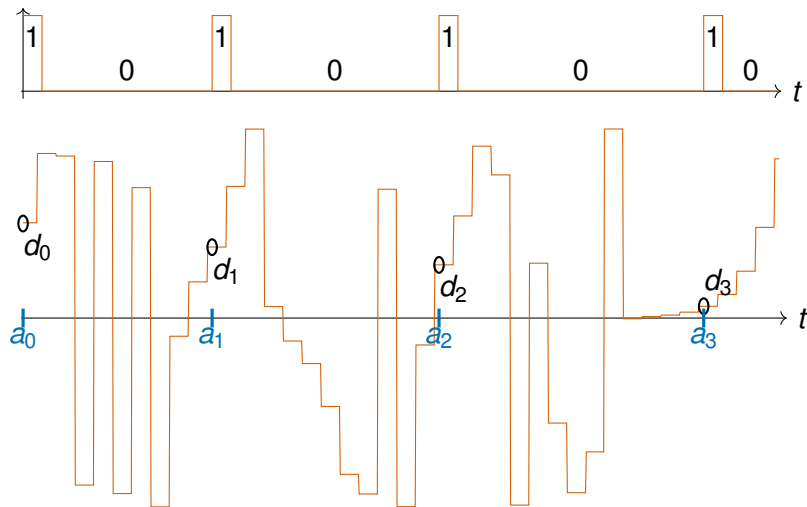


dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j < i} d_j$

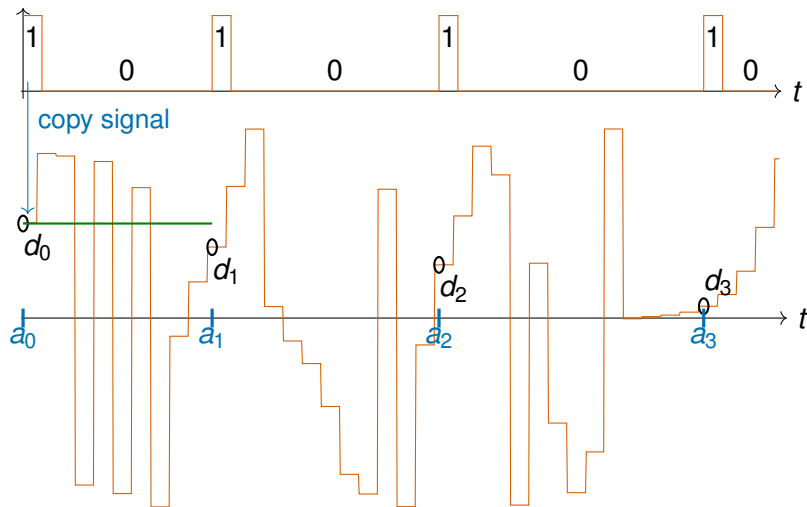
$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4, \gamma)})$$

round is the mysterious rounding function...

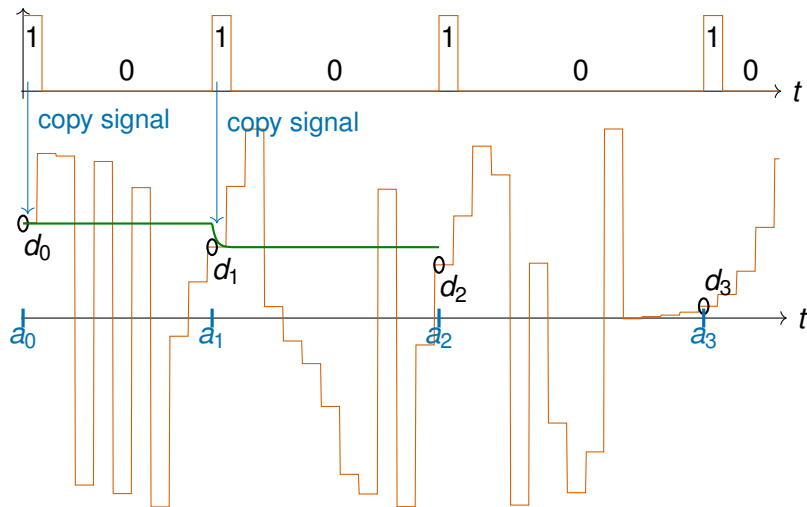
A less simplified proof



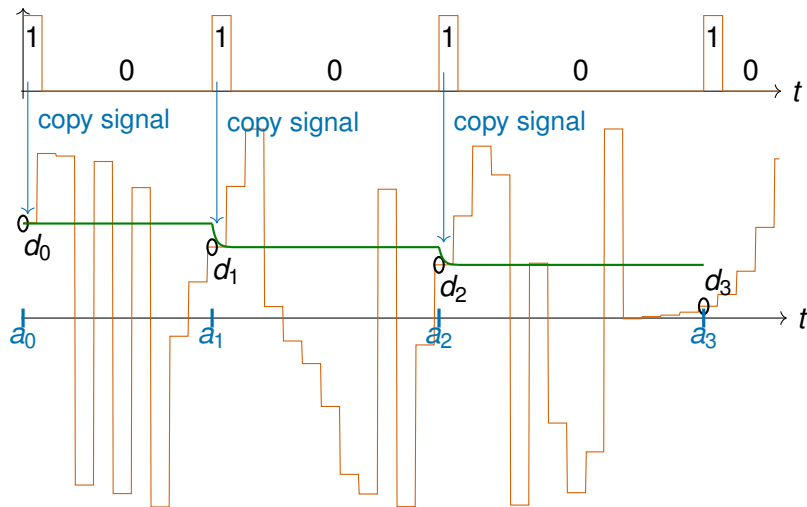
A less simplified proof



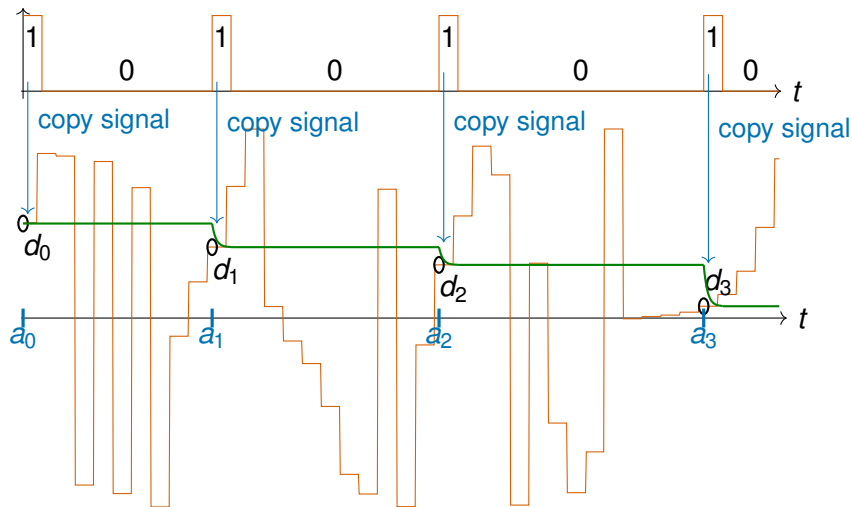
A less simplified proof



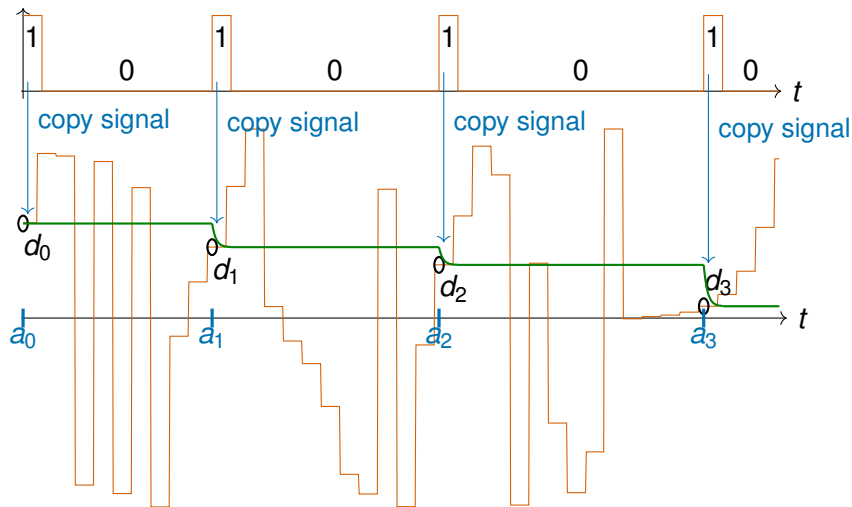
A less simplified proof



A less simplified proof

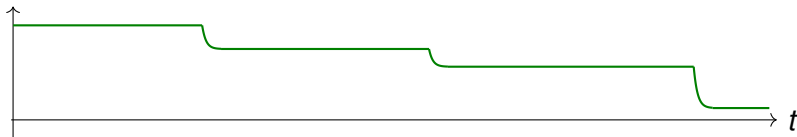


A less simplified proof



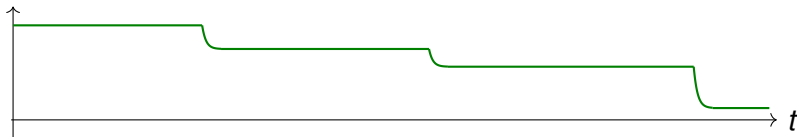
This copy operation is the “non-trivial” part.

A less simplified proof



We can do **almost piecewise constant functions...**

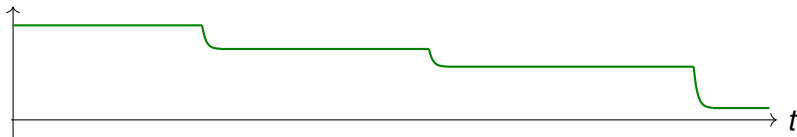
A less simplified proof



We can do **almost piecewise constant functions...**

- ▶ ...that are **bounded by 1**...
- ▶ ...and have **super slow changing frequency**.

A less simplified proof



We can do **almost piecewise constant functions...**

- ▶ ...that are **bounded by 1**...
- ▶ ...and have **super slow changing frequency**.

How do we go to arbitrarily large and growing functions? **Can a polynomial ODE even have arbitrary growth?**

An old question on growth

Building a fast-growing ODE, that exists over \mathbb{R} :

$$y_1' = y_1 \quad \leadsto \quad y_1(t) = \exp(t)$$

An old question on growth

Building a fast-growing ODE, that exists over \mathbb{R} :

$$\begin{array}{lll} y_1' = y_1 & \leadsto & y_1(t) = \exp(t) \\ y_2' = y_1 y_2 & \leadsto & y_1(t) = \exp(\exp(t)) \end{array}$$

An old question on growth

Building a fast-growing ODE, **that exists over \mathbb{R}** :

$$\begin{array}{lll} y_1' = y_1 & \rightsquigarrow & y_1(t) = \exp(t) \\ y_2' = y_1 y_2 & \rightsquigarrow & y_1(t) = \exp(\exp(t)) \\ \dots & & \dots \\ y_n' = y_1 \cdots y_n & \rightsquigarrow & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t) \end{array}$$

An old question on growth

Building a fast-growing ODE, **that exists over \mathbb{R}** :

$$\begin{array}{lll} y_1' = y_1 & \leadsto & y_1(t) = \exp(t) \\ y_2' = y_1 y_2 & \leadsto & y_1(t) = \exp(\exp(t)) \\ \dots & & \dots \\ y_n' = y_1 \cdots y_n & \leadsto & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t) \end{array}$$

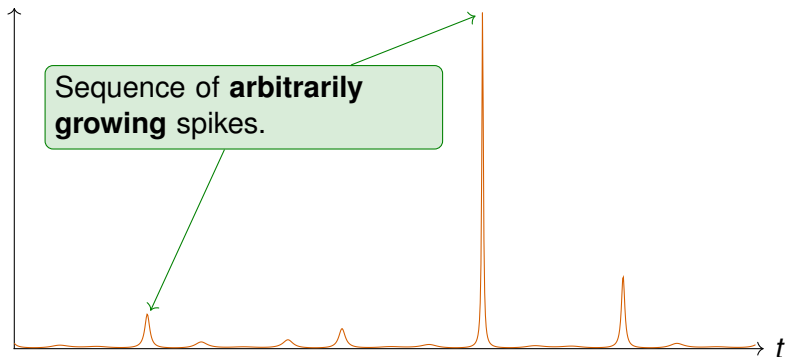
Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

An old question on growth

Counter-example (Vijayaraghavan, 1932)

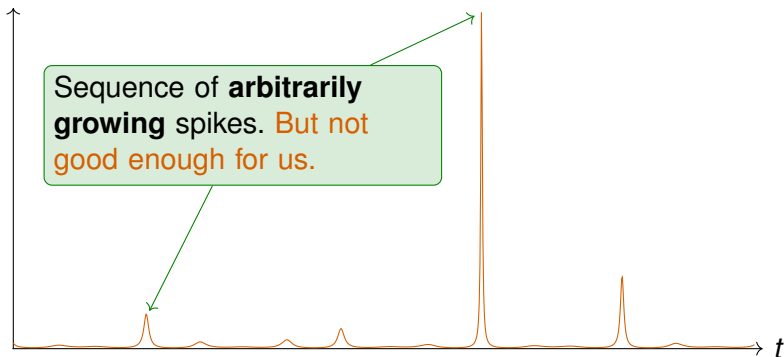
$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



An old question on growth

Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



An old question on growth

Theorem

There exists a polynomial $p : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

satisfies
$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

$$y_1(t) \geq f(t), \quad \forall t \geq 0.$$

An old question on growth

Theorem

There exists a polynomial $p : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

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$$y_1(t) \geq f(t), \quad \forall t \geq 0.$$

Note : both results require α to be **transcendental**. Conjecture still open for **rational** (or algebraic) coefficients.

Proof gem : iteration with differential equations

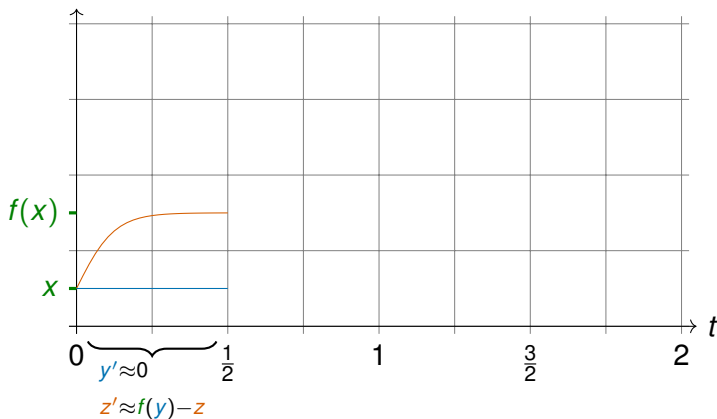
Assume f is generable, can we **iterate** f with an ODE ?

That is, build a generable y such that $y(x, n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$

Proof gem : iteration with differential equations

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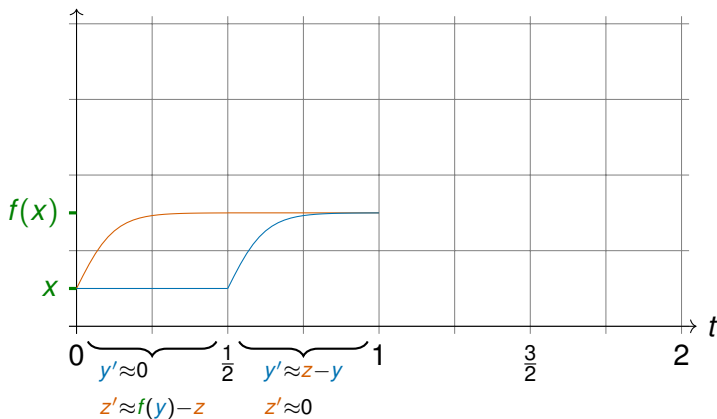
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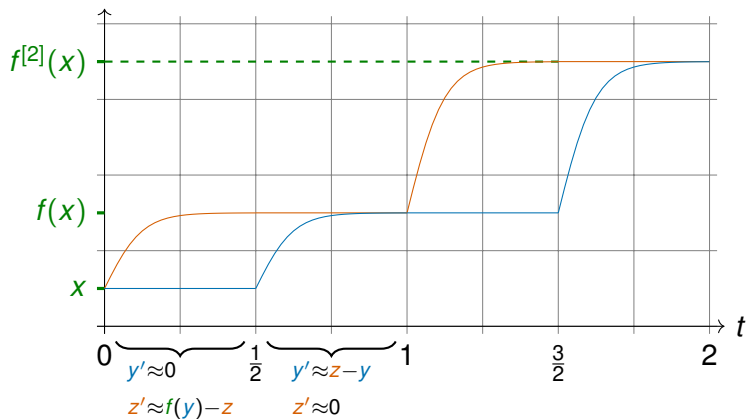
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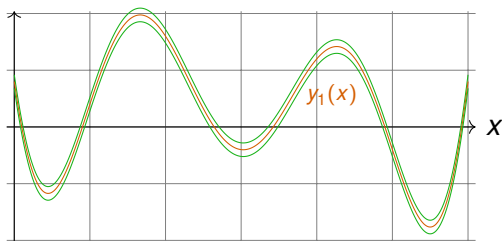
Proof gem : iteration with differential equations

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That is, build a generable y such that $y(x, n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$



Universal initial value problem (IVP)



Notes :

- ▶ **system** of ODEs,
- ▶ y is analytic,
- ▶ we need $d \approx 300$.

Theorem

There exists a **fixed** (vector of) polynomial p such that for any continuous functions f and ε , there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \quad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \rightarrow \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leq \varepsilon(t).$$

Remark : α is usually transcendental, but computable from f and ε

Backup slides

Generable functions (total, univariate)

Definition

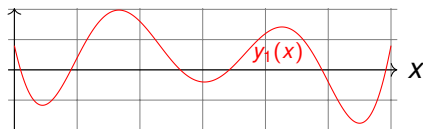
$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

Types

- ▶ $d \in \mathbb{N}$: dimension
- ▶ $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$: field
- ▶ $p \in \mathbb{K}^d[\mathbb{R}^n]$: polynomial vector (coef. in \mathbb{K})
- ▶ $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$



Note : existence and unicity of y by Cauchy-Lipschitz theorem.

Generable functions (total, univariate)

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Example : $f(x) = x$ ▶ identity

$$y(0) = 0, \quad y' = 1 \quad \leadsto \quad y(x) = x$$

Generable functions (total, univariate)

Definition

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Example : $f(x) = x^2$ ▶ squaring

$$\begin{array}{llll} y_1(0) = 0, & y_1' = 2y_2 & \leadsto & y_1(x) = x^2 \\ y_2(0) = 0, & y_2' = 1 & \leadsto & y_2(x) = x \end{array}$$

Generable functions (total, univariate)

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Example : $f(x) = x^n$ ▶ n^{th} power

$$\begin{array}{lll} y_1(0) = 0, & y'_1 = ny_2 & \leadsto y_1(x) = x^n \\ y_2(0) = 0, & y'_2 = (n-1)y_3 & \leadsto y_2(x) = x^{n-1} \\ \dots & \dots & \dots \\ y_n(0) = 0, & y'_n = 1 & \leadsto y_n(x) = x \end{array}$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

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Example : $f(x) = \exp(x)$ ▶ exponential

$$y(0) = 1, \quad y' = y \leadsto y(x) = \exp(x)$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

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Example : $f(x) = \sin(x)$ or $f(x) = \cos(x)$

▶ **sine/cosine**

$$\begin{array}{llll} y_1(0) = 0, & y_1' = y_2 & \leadsto & y_1(x) = \sin(x) \\ y_2(0) = 1, & y_2' = -y_1 & \leadsto & y_2(x) = \cos(x) \end{array}$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

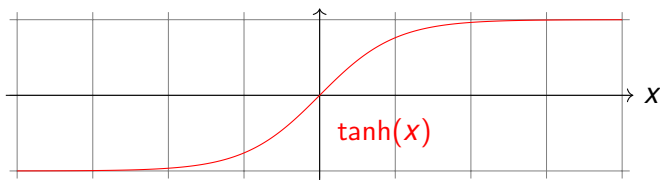
satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

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Example : $f(x) = \tanh(x)$ ▶ hyperbolic tangent

$$y(0) = 0, \quad y' = 1 - y^2 \quad \leadsto \quad y(x) = \tanh(x)$$



Generable functions (total, univariate)

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- ▶ $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

Example : $f(x) = \frac{1}{1+x^2}$ ▶ rational function

$$f'(x) = \frac{-2x}{(1+x^2)^2} = -2xf(x)^2$$

$$\begin{array}{llll} y_1(0) = 1, & y_1' = -2y_2y_1^2 & \rightsquigarrow & y_1(x) = \frac{1}{1+x^2} \\ y_2(0) = 0, & y_2' = 1 & \rightsquigarrow & y_2(x) = x \end{array}$$

Generable functions (total, univariate)

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Example : $f = g \pm h$ ▶ **sum/difference**

$$(g \pm h)' = g' \pm h'$$

assume :

$$z(0) = z_0,$$

$$z' = p(z)$$

$$\leadsto z_1 = g$$

$$w(0) = w_0,$$

$$w' = q(w)$$

$$\leadsto w_1 = h$$

then :

$$y(0) = z_{0,1} + w_{0,1},$$

$$y' = p_1(z) \pm q_1(w) \leadsto y = z_1 \pm w_1$$

Generable functions (total, univariate)

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- ▶ $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

Example : $f = gh$ ▶ **product**

$$(gh)' = g'h + gh'$$

assume :

$$z(0) = z_0,$$

$$z' = p(z)$$

$$\leadsto z_1 = g$$

$$w(0) = w_0,$$

$$w' = q(w)$$

$$\leadsto w_1 = h$$

then :

$$y(0) = z_{0,1} w_{0,1},$$

$$y' = p_1(z)w_1 + z_1q_1(w) \leadsto y = z_1w_1$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

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Example : $f = \frac{1}{g}$ ▶ inverse

$$f' = \frac{-g'}{g^2} = -g' f^2$$

assume :

$$z(0) = z_0, \quad z' = p(z) \quad \rightsquigarrow \quad z_1 = g$$

then :

$$y(0) = \frac{1}{z_{0,1}}, \quad y' = -p_1(z)y^2 \quad \rightsquigarrow \quad y = \frac{1}{z_1}$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

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- ▶ $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

Example : $f = \int g$ ▶ integral

assume :

$$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$$

then :

$$y(0) = 0, \quad y' = z_1 \quad \leadsto \quad y = \int z_1$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

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Example : $f = g'$ ▶ derivative

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

assume :

$$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$$

then :

$$y(0) = p_1(z_0), \quad y' = \nabla p_1(z) \cdot p(z) \quad \leadsto \quad y = z_1''$$

Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \quad y'(x) = p(y(x))$$

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- ▶ $y_0 \in \mathbb{K}^d, y : \mathbb{R} \rightarrow \mathbb{R}^d$

Example : $f = g \circ h$ ▶ **composition**

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

assume :

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Generable functions (total, univariate)

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is **generable** if there exists d, p and y_0 such that the solution y to

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satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

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- ▶ $d \in \mathbb{N}$: dimension
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Example : $f = g \circ h$ ▶ **composition**

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Is this coefficient in \mathbb{K} ? Fields with this property are called **generable**.

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Example : $f' = \tanh \circ f$ ▶ Non-polynomial differential equation

$$f'' = (\tanh' \circ f)f' = (1 - (\tanh \circ f)^2)f'$$

$$\begin{array}{llll} y_1(0) = f(0), & y_1' = y_2 & \leadsto & y_1(x) = f(x) \\ y_2(0) = \tanh(f(0)), & y_2' = (1 - y_2^2)y_2 & \leadsto & y_2(x) = \tanh(f(x)) \end{array}$$

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Example : $f(0) = f_0, f' = g \circ f$ ▶ Initial Value Problem (IVP)

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

assume :

$$z(0) = z_0, \quad z' = p(z) \quad \leadsto \quad z_1 = g$$

then :

$$y(0) = p_1(z_0), \quad y' = \nabla p_1(z) \cdot p(z) \quad \leadsto \quad y = z_1''$$

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **generable** if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \quad J_y(x) = p(y(x))$$

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$J_y(x)$ = Jacobian matrix of y at x

Types

- ▶ $n \in \mathbb{N}$: input dimension
- ▶ $d \in \mathbb{N}$: dimension
- ▶ $p \in \mathbb{K}^{d \times d}[\mathbb{R}^d]$: polynomial matrix
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Notes :

- ▶ Partial differential equation !
- ▶ Unicity of solution y ...
- ▶ ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2$ ($n = 2, d = 3$)

▶ **monomial**

$$y(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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Example : $f(x_1, x_2) = x_1 x_2^2$ ▶ **monomial**

$$\begin{array}{llll} y_1(0,0)=0, & \partial_{x_1} y_1 = y_2^2, & \partial_{x_2} y_1 = 2x_1 y_2 & \leadsto y_1(x) = x_1 x_2^2 \\ y_2(0,0)=0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \leadsto y_2(x) = x_1 \\ y_3(0,0)=0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \leadsto y_3(x) = x_2 \end{array}$$

This is tedious !

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Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$

▶ inverse function

$$y(1) = 1, \quad \partial_x y = -y^2 \quad \leadsto \quad y(x) = \frac{1}{x}$$

Generable functions : summary

Nice theory for the class of multivariate **generable** functions (over connected domains) :

- ▶ analytic
- ▶ contains polynomials, \sin , \cos , \tanh , \exp , ...
- ▶ stable under \pm , \times , $/$, \circ and Initial Value Problems (IVP)
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Exercise : are all analytic functions generable ? **No**
Riemann Γ and ζ are not generable.

From discrete to real computability

Computable Analysis : lift Turing computability to real numbers

[Ko, 1991 ; Weihrauch, 2000]

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Definition

$x \in \mathbb{R}$ is computable iff \exists a computable $f : \mathbb{N} \rightarrow \mathbb{Q}$ such that :

$$|x - f(n)| \leq 10^{-n} \quad n \in \mathbb{N}$$

Examples : rational numbers, π , e , ...

n	f(n)	$\pi - f(n)$
0	3	$0.14 \leq 10^{-0}$
1	3.1	$0.04 \leq 10^{-1}$
2	3.14	$0.001 \leq 10^{-2}$
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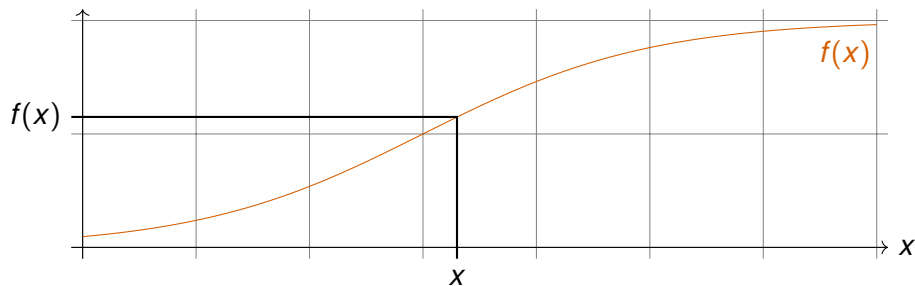
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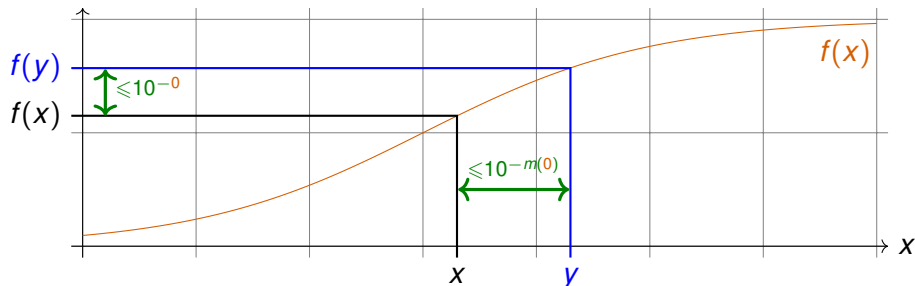
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Beware : there exists uncomputable real numbers !

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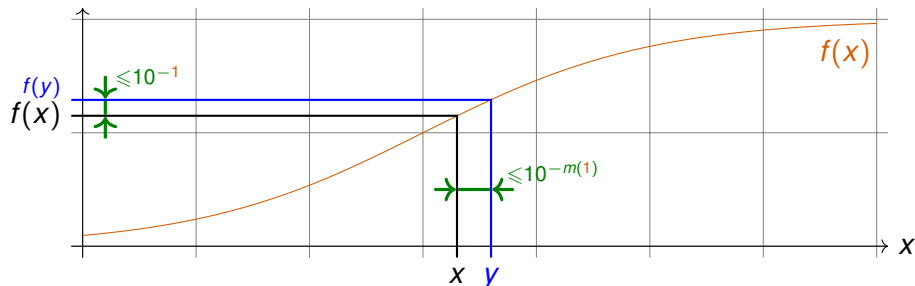
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m : modulus of continuity

From discrete to real computability



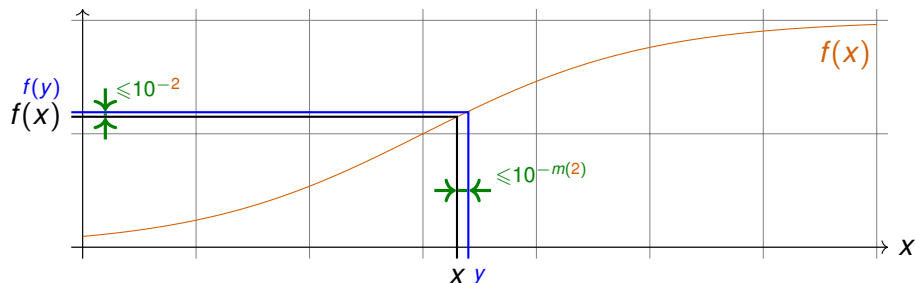
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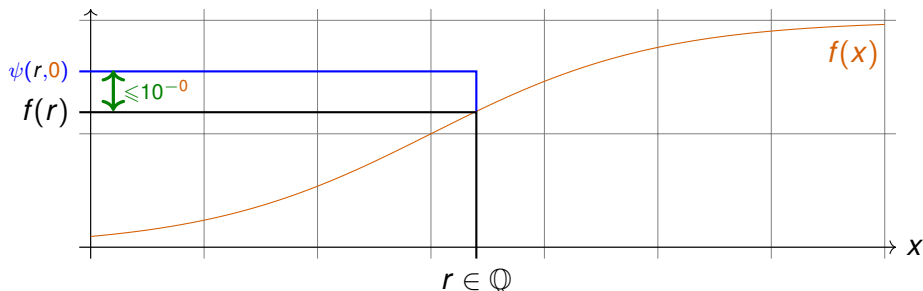
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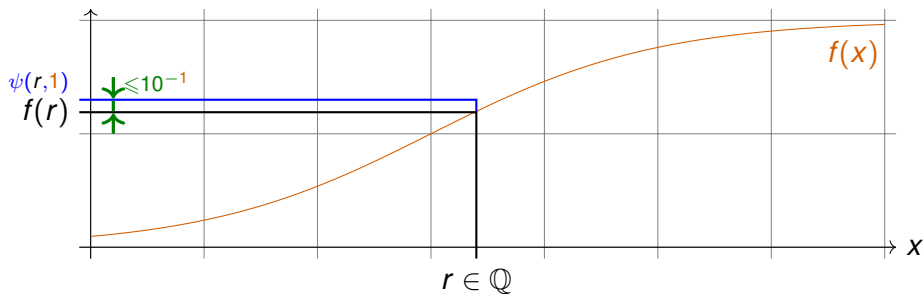
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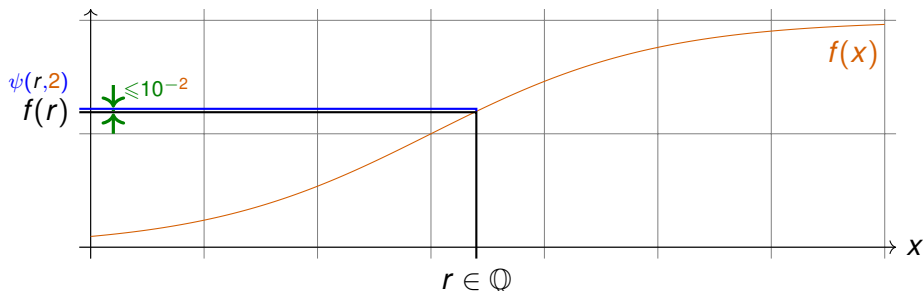
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Note : all computable functions are continuous

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Polytime complexity

Add “polynomial time computable” everywhere.

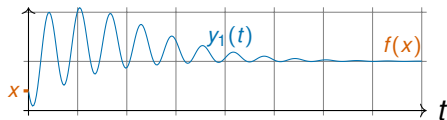
Equivalence with computable analysis

Definition (Bournez et al, 2007)

f **computable by GPAC** if $\exists p$ polynomial such that $\forall x \in [a, b]$

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$$y_1(t) \xrightarrow[t \rightarrow \infty]{} f(x)$$

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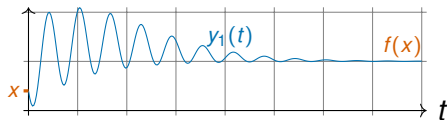
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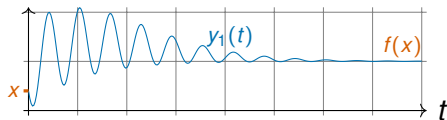
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1. In Computable Analysis, a standard model over reals built from Turing machines.